$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}}}}}}}$$

SOME CONTINUED FRACTIONS AND PROOFS

Paul van de Veen

Some Continued Fractions

and proofs

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}}}}}}}$$

Paul van de Veen
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Some Continued Fractions and proofs

Introduction

The three following expressions share a couple of properties.

$$2+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cfrac{9^2}{2+\ddots}}}}} \qquad 1+\cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{7+\cfrac{8}{9+\cfrac{10}{11+\ddots}}}}} \qquad 1+\cfrac{1^2}{1+\cfrac{2^2}{1+\cfrac{3^2}{1+\cfrac{4^2}{1+\cfrac{5^2}{1+\ddots}}}}}$$

Firstly, they are all typical examples of a “Continued Fraction”. Secondly, they all exhibit an easy to discern pattern. Thirdly, they are all difficult to evaluate with basic mathematic techniques. They simply seem to escape any form of analysis.
The fascinating aspect is that 'Continued Fractions' contain nothing more than the elementary operations of addition, subtraction, multiplication and division and still form very nice expressions for irrational numbers as roots and mathematical constants such as π and e.
In 2019, I was seized by an addiction to continued fractions.
Initially, my fascination was motivated by the desire to understand a single puzzling continued fraction. But after one came another… It turns out that there are countless continued fractions for π. Others continued fractions give nice and sometimes efficient approaches for e.
There are also continued fractions that are aesthetically very fascinating but have no answer in elementary constants. For some, special functions are indispensable, such as *Gamma*, *Beta, Bessel* and *Confluent HyperGeometric functions*.
Some continued fractions appear to have a 'mirror image' with completely different properties - unlike biological twins. A fascinating world opened up to me and without too many problems this publication could have numbered hundreds of pages.

The structure of this book is as follows:
At first an overview of all theorems. In all continued fractions presented here, the pattern of the continued fraction is always so clear that the following terms are evident without further explication.
Every continued fraction is “*pleasing for the eye*”. Statements numbered, as for example 7.1, 7.2, 7.3, belong together. Dozens more could easily have been added.
If one would stop reading after this overview, you will miss the most important part of mathematics. Proofs...

Although there is extensive literature on Continued Fractions shows that an easy accessible collection of proofs for these theorems is absent. Proofs are usually deeply hidden in the books and texts of for example Leonard Euler and others. Some theorems are solely proven in recent articles, but proofs of the general case are still difficult to find. Which is why I decided to summarize and revitalize proofs of a large collection of Continued Fractions.
Some theorems are easy to prove with some elementary algebra, others can be proven relatively easily by making clever use of series and or recursion. But where one technique provides excellent service for a certain group of continued fractions, a completely different approach is required for another group.
Some continued fractions I could only prove by using the somewhat complicated "*Euler's Differential Method*" in which a recursion leads to a differential equation that, after solution, leads to the quotient of two integrals through which the continued fraction is determined.
In some proofs the *Bessel* functions of the 1st and 2nd order and the *Hypergeometric* functions of Gauss make their appearance. But the vast majority of my proofs are "*Eulerian*". These proofs use the mathematics that Euler had to his disposal.

Obviously, any fraction can be expressed as a finite continued fraction through a simple principle of division and inversion.
For example $\frac{62}{47}=1+\frac{15}{47}=1+\frac{1}{\frac{47}{15}}=1+\frac{1}{3+\frac{2}{15}}=1+\frac{1}{3+\frac{1}{\frac{15}{2}}}=1+\frac{1}{3+\frac{1}{7+\frac{1}{2}}}$
Conversely, any finite continued fraction can be converted into a regular fraction by working out from back to front.

Much more interesting are the infinite ending continued fractions.
One of the best known is the continued fraction for the Golden Ratio $\phi=\frac{1}{2}\left(1+\sqrt{5}\right)$
It is defined as the ratio of the sides in a rectangle chosen such that the long side/short side ratio is equal to the *long+short side/long side* ratio.
That definition leads to: $\phi=\frac{long}{short}=\frac{long+short}{long}\Rightarrow\phi=1+\frac{1}{\phi}$
So ϕ can be written as the beautiful continued fraction on the front page.

Continued fractions became popular in the 17th century after Lord Brouncker postulated the very first (which yields π). Analyses of Leonard Euler in particular followed soon after. Continued Fractions were also used at that time as a quick calculation for a number of decimals of π en e
Using Continued Fractions, Euler and Lambert proved the irrationality of π and e.
Applications were found in the construction of more accurate calendars and in the construction of planetariums where gears with very special ratios are required.
But even without application, these continued fractions continued to fascinate.
The 19th century is considered the Golden Age for the Continued Fractions.
Great mathematicians such as Gauss, Perron, Stieltjes and Ramanujan have made important contributions. My modest contribution is that I have proven a number of theorems again and maybe one or two for the first time. I am also proud to present a new theorem for higher order roots.
The story ends with a continued fraction with such a simple structure that the demand for an answer in elementary constants or elementary functions is irresistible.
At the same time, this continued fracture stubbornly resists any attempt to penetrate it! Thousands of decimal places have been calculated! But at the same time, every attempt at analysis to date has been without any result. The wait is for someone to bring light into the darkness with a clever recursion, an ingenious sequence or a refined integral…
To quote Jacob Bernoulli, who wrote in 1689 after years of searching for the solution to a similar problem:

"*If anyone finds and communicates to us that which thus far has eluded our efforts, great will be our gratitude*"

Paul van de Veen
december 2022

An overview of all theorems:

Some Continued Fractions for second order roots

Theorem

1.1
page 19

$$2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}}}=\sqrt{2}+1$$

1.2
page 19

$$2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\ddots}}}}}=\sqrt{3}+1$$

1.3
page 19

$$4+\cfrac{4}{4+\cfrac{4}{4+\cfrac{4}{4+\cfrac{4}{4+\cfrac{4}{4+\ddots}}}}}=\sqrt{8}+2$$

1.4
page 19

$$6+\cfrac{6}{6+\cfrac{6}{6+\cfrac{6}{6+\cfrac{6}{6+\cfrac{6}{6+\ddots}}}}}=\sqrt{15}+3$$

Some Continued Fractions for higher order roots

Theorem

2.1
page 21

$$7+\cfrac{-\frac{2}{1}\cdot 4}{7+\cfrac{-\frac{5}{2}\cdot 4}{7+\cfrac{-\frac{8}{3}\cdot 4}{7+\cfrac{-\frac{11}{4}\cdot 4}{7+\cfrac{-\frac{14}{5}\cdot 4}{7+\ddots}}}}}=4\sqrt[3]{4}-1$$

2.2
page 21

$$9+\cfrac{-\frac{3}{1}\cdot 5}{9+\cfrac{-\frac{7}{2}\cdot 5}{9+\cfrac{-\frac{11}{3}\cdot 5}{9+\cfrac{-\frac{14}{4}\cdot 5}{9+\cfrac{-\frac{17}{5}\cdot 5}{9+\ddots}}}}}=5\sqrt[4]{5}-1$$

2.3
page 21

$$11+\cfrac{-\frac{4}{1}\cdot 6}{11+\cfrac{-\frac{9}{2}\cdot 6}{11+\cfrac{-\frac{14}{3}\cdot 6}{11+\cfrac{-\frac{19}{4}\cdot 6}{11+\cfrac{-\frac{24}{5}\cdot 6}{11+\ddots}}}}}=6\sqrt[5]{6}-1$$

3.1
page 22

$$1\cdot 9+\cfrac{1-3^2}{3\cdot 9+\cfrac{1-6^2}{5\cdot 9+\cfrac{1-9^2}{7\cdot 9+\cfrac{1-12^2}{9\cdot 9+\cfrac{1-15^2}{11\cdot 9+\ddots}}}}}=\frac{\sqrt[3]{2}+1}{\sqrt[3]{2}-1}$$

3.2
page 22

$$1\cdot 6+\cfrac{1-3^2}{3\cdot 6+\cfrac{1-6^2}{5\cdot 6+\cfrac{1-9^2}{7\cdot 6+\cfrac{1-12^2}{9\cdot 6+\cfrac{1-15^2}{11\cdot 6+\ddots}}}}}=\frac{\sqrt[3]{3}+1}{\sqrt[3]{3}-1}$$

3.3
page 22

$$1\cdot 5+\cfrac{1-3^2}{3\cdot 5+\cfrac{1-6^2}{5\cdot 5+\cfrac{1-9^2}{7\cdot 5+\cfrac{1-12^2}{9\cdot 5+\cfrac{1-15^2}{11\cdot 5+\ddots}}}}}=\frac{\sqrt[3]{4}+1}{\sqrt[3]{4}-1}=$$

3.4
page 22

$$1\cdot 8+\cfrac{1-7^2}{3\cdot 8+\cfrac{1-14^2}{5\cdot 8+\cfrac{1-21^2}{7\cdot 8+\cfrac{1-28^2}{9\cdot 8+\cfrac{1-35^2}{11\cdot 8+\ddots}}}}}=\frac{\sqrt[7]{15}+1}{\sqrt[7]{15}-1}$$

3.5
page 22

$$1\cdot 12+\cfrac{1-8^2}{3\cdot 12+\cfrac{1-16^2}{5\cdot 12+\cfrac{1-24^2}{7\cdot 12+\cfrac{1-32^2}{9\cdot 12+\cfrac{1-40^2}{11\cdot 12+\ddots}}}}}=\frac{\sqrt[8]{5}+1}{\sqrt[8]{5}-1}$$

Some Continued Fractions for π

Theorem

4.1
page 24

$$2+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cfrac{9^2}{2+\ddots}}}}}=\frac{4}{\pi}+1$$

4.2
page 29

$$6+\cfrac{1^2}{6+\cfrac{3^2}{6+\cfrac{5^2}{6+\cfrac{7^2}{6+\cfrac{9^2}{6+\ddots}}}}}=\pi+3$$

5
page 32

$$\tfrac{1}{1}+\cfrac{1}{\tfrac{1}{2}+\cfrac{1}{\tfrac{1}{3}+\cfrac{1}{\tfrac{1}{4}+\cfrac{1}{\tfrac{1}{5}+\cfrac{1}{\tfrac{1}{6}+\ddots}}}}}=\frac{2}{\pi-2}$$

6
page 37

$$1+\cfrac{1^2}{3+\cfrac{2^2}{5+\cfrac{3^2}{7+\cfrac{4^2}{9+\cfrac{5^2}{11+\ddots}}}}}=\frac{4}{\pi}$$

Theorem

7.1
page 39

$$0+\cfrac{1^2}{0+\cfrac{2^2}{0+\cfrac{3^2}{0+\cfrac{4^2}{0+\cfrac{5^2}{0+\ddots}}}}}=\frac{\pi}{2}$$

7.2
page 39

$$2+\cfrac{1^2}{2+\cfrac{2^2}{2+\cfrac{3^2}{2+\cfrac{4^2}{2+\cfrac{5^2}{2+\ddots}}}}}=\frac{2}{4-\pi}$$

7.3
page 39

$$4+\cfrac{1^2}{4+\cfrac{2^2}{4+\cfrac{3^2}{4+\cfrac{4^2}{4+\cfrac{5^2}{4+\ddots}}}}}=\frac{6}{3\pi-8}$$

7.4
page 39

$$6+\cfrac{1^2}{6+\cfrac{2^2}{6+\cfrac{3^2}{6+\cfrac{4^2}{6+\cfrac{5^2}{6+\ddots}}}}}=\frac{30}{52-15\pi}$$

7.5
page 41

$$8+\cfrac{1^2}{8+\cfrac{2^2}{8+\cfrac{3^2}{8+\cfrac{4^2}{8+\cfrac{5^2}{8+\ddots}}}}}=\frac{210}{105\pi-304}$$

Theorem

8.1
page 42

$$2+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ddots}}}}=\frac{4}{\pi}+1$$

8.2
page 42

$$6+\cfrac{1^2}{6+\cfrac{3^2}{6+\cfrac{5^2}{6+\cfrac{7^2}{6+\ddots}}}}=\pi+3$$

8.3
page 42

$$10+\cfrac{1^2}{10+\cfrac{3^2}{10+\cfrac{5^2}{10+\cfrac{7^2}{10+\ddots}}}}=\frac{16}{\pi}+5$$

8.4
page 42

$$14+\cfrac{1^2}{14+\cfrac{3^2}{14+\cfrac{5^2}{14+\cfrac{7^2}{14+\ddots}}}}=\frac{9}{4}\pi+7$$

8.5
page 42

$$18+\cfrac{1^2}{18+\cfrac{3^2}{18+\cfrac{5^2}{18+\cfrac{7^2}{18+\ddots}}}}=\frac{256}{9\pi}+9$$

8.6
page 42

$$22+\cfrac{1^2}{22+\cfrac{3^2}{22+\cfrac{5^2}{22+\cfrac{7^2}{22+\ddots}}}}=\frac{225}{64}\pi+11$$

Some Continued Fractions for *e*

Theorem

9.1
page 48

$$1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\cfrac{6}{6+\ddots}}}}}=e-1$$

9.2
page 48

$$1+\cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{2}}{1+\cfrac{\frac{1}{3}}{1+\cfrac{\frac{1}{4}}{1+\cfrac{\frac{1}{5}}{1+\ddots}}}}}=e-1$$

9.3
page 48

$$1+\cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{7+\cfrac{8}{9+\cfrac{10}{11+\ddots}}}}}=\frac{1}{\sqrt{e}-1}$$

9.4
page 48

$$1+\cfrac{6}{4+\cfrac{12}{7+\cfrac{18}{10+\cfrac{24}{13+\cfrac{30}{16+\ddots}}}}}=\frac{2}{\sqrt[3]{e^2}-1}$$

10
page 51

$$1+\cfrac{1+\cfrac{1+\cfrac{1+\cfrac{1+\cfrac{1+\cfrac{1+\iddots}{6}}{5}}{4}}{3}}{2}}{1}=e$$

Theorem

11.1
page 52

$$2+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\ddots}}}}}=\frac{e+1}{e-1}$$

11.2
page 54

$$4+\cfrac{1}{12+\cfrac{1}{20+\cfrac{1}{28+\cfrac{1}{36+\cfrac{1}{44+\ddots}}}}}=\frac{\sqrt{e}+1}{\sqrt{e}-1}$$

11.3
page 55

$$6+\cfrac{1}{18+\cfrac{1}{30+\cfrac{1}{42+\cfrac{1}{54+\cfrac{1}{66+\ddots}}}}}=\frac{\sqrt[3]{e}-1}{\sqrt[3]{e}-1}$$

11.4
page 56

$$8+\cfrac{1}{24+\cfrac{1}{40+\cfrac{1}{56+\cfrac{1}{72+\cfrac{1}{88+\ddots}}}}}=\frac{\sqrt[4]{e}+1}{\sqrt[4]{e}-1}$$

12
page 58

$$1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\ddots}}}}}=\frac{e^2+1}{e^2-1}$$

Theorem

13.1
page 61

$$\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\ddots}}}}}=\frac{1}{e-1}$$

13.2
page 61

$$1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{5+\cfrac{5}{6+\ddots}}}}}=\frac{1}{e-2}$$

13.3
page 61

$$\cfrac{2}{1+\cfrac{4}{2+\cfrac{6}{3+\cfrac{8}{4+\cfrac{10}{5+\ddots}}}}}=\frac{4}{e^2-3}$$

13.4
page 61

$$1+\cfrac{2}{2+\cfrac{4}{3+\cfrac{6}{4+\cfrac{8}{5+\cfrac{10}{6+\ddots}}}}}=\frac{4}{e^2-5}$$

Some Continued Fractions resulting in integral expressions

Theorem

14
page 63

$$1+\cfrac{1}{1+\cfrac{3}{1+\cfrac{5}{1+\cfrac{7}{1+\cfrac{9}{1+\ddots}}}}}=1+\frac{2\int\limits_0^\infty x^{\frac{1}{2}}\cdot e^{-x-x^2}dx}{\int\limits_0^\infty x^{-\frac{1}{2}}\cdot e^{-x-x^2}dx}$$

15
page 65

$$1+\cfrac{1}{1+\cfrac{2}{1+\cfrac{3}{1+\cfrac{4}{1+\cfrac{5}{1+\ddots}}}}}=\frac{1}{\sqrt{e}\int\limits_1^\infty e^{-\frac{1}{2}x^2}dx}$$

16
page 67

$$\cfrac{1}{2+\cfrac{3}{4+\cfrac{5}{6+\cfrac{7}{8+\cfrac{9}{10+\cfrac{11}{12+\ddots}}}}}}=\frac{\int\limits_0^1 x^{\frac{1}{2}}\cdot e^{\frac{1}{2}x}\,dx}{\int\limits_0^1 x^{-\frac{1}{2}}\cdot e^{\frac{1}{2}x}\,dx}$$

Some Continued Fractions resulting in special functions

Theorem

17
page 68

$$1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{6+\ddots}}}}}=\frac{I_0(2)}{I_1(2)}$$

18
page 75

$$1+\cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{3}}{1+\cfrac{\frac{1}{5}}{1+\cfrac{\frac{1}{7}}{1+\cfrac{\frac{1}{9}}{1+\ddots}}}}}=\cfrac{2}{1+\cfrac{I_1\left(\frac{1}{4}\right)}{I_0\left(\frac{1}{4}\right)}}$$

19
page 78

$$\frac{1}{1}+\cfrac{1}{\frac{1}{3}+\cfrac{1}{\frac{1}{5}+\cfrac{1}{\frac{1}{7}+\cfrac{1}{\frac{1}{9}+\cfrac{1}{\frac{1}{11}+\ddots}}}}}=\frac{\frac{1}{8}\left(\frac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})}\right)^2+1}{\frac{1}{8}\left(\frac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})}\right)^2-1}$$

20
page 83

$$1+\cfrac{1^2}{1+\cfrac{2^2}{1+\cfrac{3^2}{1+\cfrac{4^2}{1+\cfrac{5^2}{1+\ddots}}}}}=\frac{1}{\ln(2)}$$

21
page 85

$$1^2+\cfrac{1}{2^2+\cfrac{1}{3^2+\cfrac{1}{4^2+\cfrac{1}{5^2+\cfrac{1}{6^2+\ddots}}}}}=\ ?$$

Some ‘Mirrored’ Continued Fractions

Theorem

5 & 9.2
page 32 & 48

$$\frac{1}{1}+\cfrac{1}{\frac{1}{2}+\cfrac{1}{\frac{1}{3}+\cfrac{1}{\frac{1}{4}+\cfrac{1}{\frac{1}{5}+\cfrac{1}{\frac{1}{6}+\ddots}}}}} \leftrightarrow 1+\cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{2}}{1+\cfrac{\frac{1}{3}}{1+\cfrac{\frac{1}{4}}{1+\cfrac{\frac{1}{5}}{1+\ddots}}}}}$$

12 & 14
page 58 & 63

$$1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\ddots}}}}} \leftrightarrow 1+\cfrac{1}{1+\cfrac{3}{1+\cfrac{5}{1+\cfrac{7}{1+\cfrac{9}{1+\ddots}}}}}$$

15 & 17
page 65 & 68

$$1+\cfrac{1}{1+\cfrac{2}{1+\cfrac{3}{1+\cfrac{4}{1+\cfrac{5}{1+\ddots}}}}} \leftrightarrow 1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{6+\ddots}}}}}$$

18 & 19
page 75 & 78

$$1+\cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{3}}{1+\cfrac{\frac{1}{5}}{1+\cfrac{\frac{1}{7}}{1+\cfrac{\frac{1}{9}}{1+\frac{1}{11}\ddots}}}}} \leftrightarrow \frac{1}{1}+\cfrac{1}{\frac{1}{3}+\cfrac{1}{\frac{1}{5}+\cfrac{1}{\frac{1}{7}+\cfrac{1}{\frac{1}{9}+\cfrac{1}{\frac{1}{11}+1\ddots}}}}}$$

Theorem

9.3 & 16
page 48 & 67

$$1+\cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{7+\cfrac{8}{9+\cfrac{10}{11+\ddots}}}}} \leftrightarrow 0+\cfrac{1}{2+\cfrac{3}{4+\cfrac{5}{6+\cfrac{7}{8+\cfrac{9}{10+\ddots}}}}}$$

20 & 21
page 83 & 85

$$1+\cfrac{1^2}{1+\cfrac{2^2}{1+\cfrac{3^2}{1+\cfrac{4^2}{1+\cfrac{5^2}{1+\ddots}}}}} \leftrightarrow 1^2+\cfrac{1}{2^2+\cfrac{1}{3^2+\cfrac{1}{4^2+\cfrac{1}{5^2+\cfrac{1}{6^2+\ddots}}}}}$$

Proofs

Some Continued Fractions for second order roots

Theorem 1.1

$$\sqrt{2}+1=2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}}}$$

Theorem 1.2

$$\sqrt{3}+1=2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\ddots}}}}}$$

Theorem 1.3

$$\sqrt{8}+2=4+\cfrac{4}{4+\cfrac{4}{4+\cfrac{4}{4+\cfrac{4}{4+\cfrac{4}{4+\ddots}}}}}$$

Theorem 1.4

$$\sqrt{15}+3=6+\cfrac{6}{6+\cfrac{6}{6+\cfrac{6}{6+\cfrac{6}{6+\cfrac{6}{6+\ddots}}}}}$$

Any square root can be written as an infinite regular continued fraction.
The proof is the following simple algebraic identity.

$$\left(\sqrt{n}-a\right)\cdot\left(\sqrt{n}+a\right)=n-a^2 \quad\Rightarrow \sqrt{n}=a+\frac{n-a^2}{a+\sqrt{n}} \quad\Rightarrow \sqrt{n}+a=2a+\cfrac{n-a^2}{2a+\cfrac{n-a^2}{2a+\cfrac{n-a^2}{2a+\cfrac{n-a^2}{2a+\ddots}}}}$$

Theorem 1.1, 1.2, 1.3 and 1.4 are direct examples.

Conversely, any infinite, regular continued fraction is a representation of a square root (or a *radical* like $a+\sqrt{b}$) because

$$x = a + \cfrac{b}{a + \cfrac{b}{a + \ddots}} \quad \Rightarrow \quad x = a + \frac{b}{x} \quad \Rightarrow \quad x^2 - ax - b = 0 \quad \Rightarrow x = \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 + b}$$

A useful notation for infinite continued fractions is the "*Kettungbrüch*" notation:

$$b_0 + \operatorname*{K}_{n=1}^{\infty}\left(\tfrac{a_k}{b_k}\right) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ddots}}}}$$

The examples given above of continued fractions in this notation:

$$\phi = 1 + \operatorname*{K}_{n=1}^{\infty}\left(\tfrac{1}{1}\right)$$

$$\sqrt{2} + 1 = 2 + \operatorname*{K}_{n=1}^{\infty}\left(\tfrac{1}{2}\right)$$

$$\sqrt{8} + 2 = 4 + \operatorname*{K}_{n=1}^{\infty}\left(\tfrac{4}{4}\right)$$

$$\sqrt{15} + 3 = 6 + \operatorname*{K}_{n=1}^{\infty}\left(\tfrac{6}{6}\right)$$

Some Continued Fractions for higher order roots

Higher order roots do not satisfy an identity as the square roots and therefore cannot be written as infinitely regular Continued Fractions. By 'regular' it is then meant that the continued fraction only contains constants. But 'regularity' can also mean that the continued fraction follows a simple regular pattern.
Seen in this way, higher roots of power can indeed be written as an infinite regular continued fraction.
Examples are theorems 2.1 and 2.2, which are generalized in theorem 2.3.

Theorem 2.1

$$7+\cfrac{-\frac{2}{1}\cdot 4}{7+\cfrac{-\frac{5}{2}\cdot 4}{7+\cfrac{-\frac{8}{3}\cdot 4}{7+\cfrac{-\frac{11}{4}\cdot 4}{7+\cfrac{-\frac{14}{5}\cdot 4}{7+\ddots}}}}} = 4\sqrt[3]{4}-1$$

Theorem 2.2

$$11+\cfrac{-\frac{4}{1}\cdot 6}{11+\cfrac{-\frac{9}{2}\cdot 6}{11+\cfrac{-\frac{14}{3}\cdot 6}{11+\cfrac{-\frac{19}{4}\cdot 6}{11+\cfrac{-\frac{24}{5}\cdot 6}{11+\ddots}}}}} = 6\sqrt[5]{6}-1$$

Theorem 2.3

$$(1+p)\sqrt[p]{p+1}-1 = 1+2p+\operatorname*{K}_{n=0}^{\infty}\left(\frac{(1+p)\cdot\frac{(1-(n+1)p)}{n+1}}{1+2p}\right)$$

The proof of the general theorem 2.3 is given in lemma 8, page 97.
It is not simple and it applies *'Euler's Differential Method'*.

Whether the examples given above are “*beautiful*” is a matter of taste.
There is certainly a lot of regularity.
It is obvious how to continue these continued fractions.

Theorem 3.1

$$1\cdot 9+\cfrac{1-3^2}{3\cdot 9+\cfrac{1-6^2}{5\cdot 9+\cfrac{1-9^2}{7\cdot 9+\cfrac{1-12^2}{9\cdot 9+\ddots}}}}=\frac{\sqrt[3]{2}+1}{\sqrt[3]{2}-1}$$

Theorem 3.2

$$1\cdot 6+\cfrac{1-3^2}{3\cdot 6+\cfrac{1-6^2}{5\cdot 6+\cfrac{1-9^2}{7\cdot 6+\cfrac{1-12^2}{9\cdot 6+\ddots}}}}=\frac{\sqrt[3]{3}+1}{\sqrt[3]{3}-1}$$

Theorem 3.3

$$1\cdot 5+\cfrac{1-3^2}{3\cdot 5+\cfrac{1-6^2}{5\cdot 5+\cfrac{1-9^2}{7\cdot 5+\cfrac{1-12^2}{9\cdot 5+\ddots}}}}=\frac{\sqrt[3]{4}+1}{\sqrt[3]{4}-1}$$

Theorem 3.4

$$1\cdot 8+\cfrac{1-7^2}{3\cdot 8+\cfrac{1-14^2}{5\cdot 8+\cfrac{1-21^2}{7\cdot 8+\cfrac{1-28^2}{9\cdot 8+\ddots}}}}=\frac{\sqrt[7]{15}+1}{\sqrt[7]{15}-1}$$

Theorem 3.5

$$1\cdot 12+\cfrac{1-8}{3\cdot 12+\cfrac{1-16^2}{5\cdot 12+\cfrac{1-24^2}{7\cdot 12+\cfrac{1-32^2}{9\cdot 12+\ddots}}}}=\frac{\sqrt[8]{5}+1}{\sqrt[8]{5}-1}$$

Theorem 3.1 to 3.5 show that also higher order roots can be written as a continued fraction. A general method was described by M. Sardina in 2007.
Based on the Sardina approach, I was able to derive an elegant Continued Fraction for higher order roots. Maybe this should be called "*Van de Veen's formula for higher order roots*". See lemma 9, page 102.

Theorem 3.1 t/m 3.5 are all examples of the following theorem.

$$\frac{\sqrt[n]{A}+1}{\sqrt[n]{A}-1}=1\cdot C+\cfrac{1-(1\cdot n)^2}{3\cdot C+\cfrac{1-(2\cdot n)^2}{5\cdot C+\cfrac{1-(3\cdot n)^2}{7\cdot C+\cfrac{1-(4\cdot n)^2}{9\cdot C+\ddots}}}}\quad\text{met}\quad C=n\cdot\frac{A+1}{A-1}$$

Choosing a specific *n* and *A* determines *C* and therefore determines the Continued Fraction for $\sqrt[n]{A}$

In Theorem 3.1: $n=3,\quad A=2,\quad C=3\cdot\frac{2+1}{1}=9$
In Theorem 3.2: $n=3,\quad A=3,\quad C=3\cdot\frac{3+1}{2}=6$
In Theorem 3.3: $n=5,\quad A=11,\quad C=5\cdot\frac{11+1}{10}=6$
In Theorem 3.4: $n=7,\quad A=15,\quad C=7\cdot\frac{15+1}{14}=8$
In Theorem 3.5: $n=8,\quad k=2, A=1+\frac{n}{k}=5,\quad C=k(A+1)=12$

When we define a “*nice-looking*” Continued Fraction as containing only integers in its expansion, *C* must be an integer. But this restricts possible Continued Fractions to a small collection of five different groups.

$A=2$	$\Rightarrow C=3n$	The group Continued Fractions for $\sqrt[n]{2}$
$A=3$	$\Rightarrow C=2n$	The group Continued Fractions for $\sqrt[n]{3}$
$A=n+1$	$\Rightarrow C=n+2$	The group Continued Fractions for $\sqrt[n]{n+1}$
$A=2n+1$	$\Rightarrow C=n+1$	The group Continued Fractions for $\sqrt[n]{2n+1}$
$A=1+\frac{n}{k}$	$\Rightarrow C=k(A+1)$	The group of Continued Fractions for $\sqrt[n]{\frac{n}{k}+1}$

The regularity that occurs is fascinating. Yet….
These theorems miss something important in mathematics. Elegance.

Some continued fractions for π

Theorem 4.1

$$\frac{4}{\pi}+1=2+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ddots}}}}$$

This is one of the very first Continued Fractions. William Brouncker published it in 1656 without proof. That in itself is remarkable because the theorem can be proved in many ways.

Proof 4.1-a makes use of Euler´s Continued Fraction formula, lemma 1, page 86, applied to the integral $\int_0^1 \frac{x^{n-1}}{1+x^m}dx$ and expanded in a series, lemma 3, page 89.

Proof 4.1-b uses a slight variation of Euler's Continued Fraction formula, lemma 2, page 88, applied to the Gregory-Leibniz series.

Proof 4.1-c makes use of a recursion property for the series: $I_N = \sum_{k=0}^{N} \frac{(-1)^k}{(2k+1)} - \frac{\pi}{4}$

Proof 4.1-d uses the recursion property of the integral $I_N = \int_0^1 \frac{x^{2N}}{1+x^2}dx$

Proof 4.1-a

A proof using Euler´s Continued Fraction formula lemma 3, page 89:

$$\int_0^1 \frac{x^{n-1}}{1+x^m}dx = \cfrac{1}{n+\cfrac{n^2}{m+\cfrac{(m+n)^2}{m+\cfrac{(2m+n)^2}{m+\cfrac{(3m+n)^2}{m+\ddots}}}}}$$

With $m=2$ and $n=1$ it follows that

$$\int_0^1 \frac{1}{1+x^2}dx = \arctan(x)\Big|_0^1 = \frac{\pi}{4} = \cfrac{1}{1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ddots}}}}}$$

With a little algebraic manipulation this can be rewritten as

$$\frac{4}{\pi}+1 = 2+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ddots}}}}$$

Proof 4.1-b

From the series expansion

$$\arctan(x)=\int\frac{1}{1+x^2}dx=\int 1-x^2+x^4-x^6+\ldots dx=x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7+\ldots$$

follows immediately with $x=1$ the Gregory-Leibniz series: $\dfrac{\pi}{4}=\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)}=\dfrac{1}{1}-\dfrac{1}{3}+\dfrac{1}{5}\ldots$

Now we apply Euler's Continued Fraction formula lemma 2, page 88 on this series.
With $b_0=1,\quad b_1=3,\quad b_2=5$ enz.. it follows that

$$\frac{\pi}{4}=\frac{1}{b_0}-\frac{1}{b_1}+\frac{1}{b_2}-\frac{1}{b_3}\ldots..=\cfrac{1}{1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ddots}}}}}\quad\Rightarrow\quad\frac{4}{\pi}+1=2+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ddots}}}}$$

Proof 4.1-c

A proof using the recursion property of a specific series.

Let: $I_N \quad = \sum_{k=0}^{N} \frac{(-1)^k}{(2k+1)} - \frac{\pi}{4}$ than $I_{N+1} = \sum_{n=0}^{N+1} \frac{(-1)^k}{(2k+1)} - \frac{\pi}{4} = I_N + \frac{(-1)^{N+1}}{2N+3}$

From this follows $I_{N+1} - I_N = \frac{(-1)^{N+1}}{2N+3} \Rightarrow I_N - I_{N-1} = \frac{(-1)^N}{2N+1} \Rightarrow \frac{I_{N+1} - I_N}{I_N - I_{N-1}} = -\frac{2N+1}{2N+3}$

Through cross multiplication and rearranging we achieve:

$(2N+3)I_{N+1} = 2I_N + (2N+1)I_{N-1}$

Dividing by I_N results in $(2N+3)\frac{I_{N+1}}{I_N} = 2 + (2N+1)\frac{I_{N-1}}{I_N}$

Now let $r_N = \frac{I_N}{I_{N-1}} \quad \Rightarrow \quad -2 + (2N+3)r_{N+1} = \frac{2N+1}{r_N} \quad \Rightarrow \quad r_N = \frac{2N+1}{-2+(2N+3)r_{N+1}} \Rightarrow$

$$r_1 = \frac{3}{-2+5r_2} = \frac{3}{-2+\frac{5^2}{-2+\frac{7^2}{-2+\ddots}}} \Rightarrow \quad r_1 = \frac{I_1}{I_0} = \frac{\frac{1}{1}-\frac{1}{3}-\frac{\pi}{4}}{\frac{1}{1}-\frac{\pi}{4}} = \frac{\frac{2}{3}-\frac{\pi}{4}}{1-\frac{\pi}{4}} = 1 + \frac{\frac{1}{3}}{\frac{\pi}{4}-1}$$

And after some small algebraic manipulations we get $\frac{4}{\pi} + 1 = 2 + \cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ddots}}}}$

Proof 4.1-d
A proof using a recursive property of a specific integral:

Let: $I_N = \int_0^1 \frac{x^{2N}}{1+x^2}dx$

Than $I_{N+1} = \int_0^1 \frac{x^{2N+2}}{1+x^2}dx = \int_0^1 \frac{(1+x^2)x^{2N} - x^{2N}}{1+x^2}dx = \frac{1}{2N+1} - I_N \quad \Rightarrow I_{N+1} + I_{N+2} = \frac{1}{2N+3}$

From this it follows that $\frac{I_N + I_{N+1}}{I_{N+1} + I_{N+2}} = \frac{2N+3}{2N+1} \Rightarrow (2N+1)I_N = 2I_{N+1} + (2N+3)I_{N+2}$

Let again $r_n = \frac{I_{N+1}}{I_N}$

Again we have a recurrence relation $2 + (2N+3)r_{n+1} = \frac{2N+1}{r_n}$ which can be easily rewritten as a Continued Fraction

$$r_n = \frac{2N+1}{2+(2N+3)r_{n+1}} \Rightarrow r_0 = \cfrac{1}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ddots}}}}$$

$$r_0 = \frac{I_1}{I_0} = \frac{\int_0^1 \frac{x^2}{1+x^2}dx}{\int_0^1 \frac{1}{1+x^2}dx} = \frac{1-\frac{\pi}{4}}{\frac{\pi}{4}} = \frac{4}{\pi} - 1 = \cfrac{1}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ddots}}}} \Rightarrow$$

$$\frac{4}{\pi} + 1 = 2 + \cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ddots}}}}$$

Theorem 4.2

$$\pi+3=6+\cfrac{1^2}{6+\cfrac{3^2}{6+\cfrac{5^2}{6+\cfrac{7^2}{6+\ddots}}}}$$

This is the Continued Fraction of Euler (1748), in 1999 rediscovered by L.J. Lange. Leonard Euler (1707–1783) was one of the first mathematicians who studied Continued Fractions, for example in his "*Introductio in Analysin Infinitorum*" (1748).

Proof 4-2-a uses a slight variation of *Euler's Continued Fraction formula* (see lemma 2, page 88) applied to the series $S=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n(2n+1)(2n+2)}$

Still it requires a tricky '*equivalence transformation*' to bring the result in the form of theorem 4.2.

Proof 4-2-b uses a recursion property of the series $I_N=\sum_{k=1}^{N}\frac{(-1)^{k-1}}{2k(2k+1)(2k+2)}-\left(\frac{\pi-3}{4}\right)$

Proof 4-2-a

A proof using a series:

Writing down successive terms of the series $S=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n(2n+1)(2n+2)}$ results in

$$S=\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}-\frac{4\cdot(-1)^{n-1}}{2n+1}+\frac{(-1)^{n-1}}{n+1}\Rightarrow$$

$$\frac{1}{4}\left[\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\ldots\right)+4\left(-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right)+\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\ldots\right)\right]=$$

$$\frac{1}{4}\left[1+4\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\ldots\right)-4\right]=\frac{\pi-3}{4}$$

But also, $S=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n(2n+1)(2n+2)}=\frac{1}{2\cdot3\cdot4}-\frac{1}{4\cdot5\cdot6}+\frac{1}{6\cdot7\cdot8}-\frac{1}{8\cdot9\cdot10}+\ldots$

Euler's Continued Fraction Formula lemma 2, page 88 with
$b_0=2\cdot3\cdot4,\ b_1=4\cdot5\cdot6,\ b_2=6\cdot7\cdot8,\ b_3=8\cdot9\cdot10$ etc. brings the Continued Fraction:

$$S=\cfrac{1}{2\cdot3\cdot4+\cfrac{(2\cdot3\cdot4)^2}{4\cdot5\cdot6-2\cdot3\cdot4+\cfrac{(4\cdot5\cdot6)^2}{6\cdot7\cdot8-4\cdot5\cdot6+\cfrac{(6\cdot7\cdot8)^2}{8\cdot9\cdot10-6\cdot7\cdot8+\cfrac{(8\cdot9\cdot10)^2}{\ddots}}}}}=$$

$$=\cfrac{1}{24+\cfrac{2^2\cdot3^2\cdot4^2}{24\cdot2^2+\cfrac{4^2\cdot5^2\cdot6^2}{24\cdot3^2+\cfrac{6^2\cdot7^2\cdot8^2}{24\cdot4^2+\cfrac{8^2\cdot9^2\cdot10^2}{\ddots}}}}}$$

After a not trivial '*equivalence transformation*' we get

$$S=\cfrac{1/4}{6+\cfrac{3^2}{6+\cfrac{5^2}{6+\cfrac{7^2}{6+\cfrac{9^2}{6+\ddots}}}}}\Rightarrow\quad\pi+3=6+\cfrac{1^2}{6+\cfrac{3^2}{6+\cfrac{5^2}{6+\cfrac{7^2}{6+\cfrac{9^2}{6+\ddots}}}}}$$

Proof 4-2-b

An alternative proof applies the recursive property of series:

Let $I_N = \sum_{k=1}^{N} \frac{(-1)^{k-1}}{2k(2k+1)(2k+2)} - \left(\frac{\pi-3}{4}\right) \Rightarrow$

$$\Rightarrow I_{N+1} = \frac{1}{4}\sum_{k=1}^{N} \frac{(-1)^{k-1}}{k(2k+1)(k+1)} + \frac{(-1)^N}{(N+1)(2N+3)(N+2)} - \left(\frac{\pi-3}{4}\right)$$

$$\begin{cases} I_{N+1} - I_N = \frac{1}{4}\frac{(-1)^N}{(N+1)(2N+3)(N+2)} \\ I_N - I_{N-1} = \frac{1}{4}\frac{(-1)^{N-1}}{N(2N+1)(N+1)} \end{cases} \Rightarrow \frac{I_{N+1}-I_N}{I_N - I_{N-1}} = -\frac{N(2N+1)}{(2N+3)(N+2)}$$

Cross=multiplication results in $(2N+3)(N+2)I_{N+1} = N(2N+1)I_{N-1} + 6(N+1)I_N$

Let $w_n = (n+1)I_N \Rightarrow (2N+3)w_{N+1} = (2N+1)w_{N-1} + 6w_N \quad \Rightarrow \frac{w_N}{w_{N-1}} = \frac{2N+1}{-6+(2N+3)\frac{w_{N+1}}{w_N}}$

$$\frac{w_1}{w_0} = \frac{2I_1}{I_0} = \cfrac{3}{-6+\cfrac{5^2}{-6+\cfrac{7^2}{-6+\ddots}}} = \cfrac{-3}{6+\cfrac{5^2}{6+\cfrac{7^2}{6+\ddots}}}$$

From the series follows $I_0 = -\frac{\pi-3}{4}$ and $I_1 = \frac{1}{2\cdot3\cdot4} - \frac{\pi-3}{4}$ so

$$\frac{w_1}{w_0} = \frac{2I_1}{I_0} = 2\cdot\frac{\left(\frac{1}{2\cdot3\cdot4} - \frac{\pi-3}{4}\right)}{\left(-\frac{\pi-3}{4}\right)} = 2 - \frac{1}{3\cdot(\pi-3)} = \cfrac{-3}{6+\cfrac{5^2}{6+\cfrac{7^2}{6+\ddots}}} \Rightarrow$$

After some simple algebraic manipulations it follows that $\pi + 3 = 6 + \cfrac{1^2}{6+\cfrac{3^2}{6+\cfrac{5^2}{6+\cfrac{7^2}{6+\ddots}}}}$

Theorem 5

$$\frac{1}{1}+\cfrac{1}{\frac{1}{2}+\cfrac{1}{\frac{1}{3}+\cfrac{1}{\frac{1}{4}+\cfrac{1}{\frac{1}{5}+\ddots}}}}=\frac{2}{\pi-2}$$

This is sometimes called the "*Harmonic*" Continued Fraction for π.
It is a very beautiful expression with a surprising result.

Proof 5.1-a uses the integral $I(n)=\int_0^{\frac{1}{4}\pi}\frac{(\cos(2x))^n}{\cos^2(x)}dx$ and the recursive property

$$I(n)=n\cdot I(n-1)-n\cdot I(n+1)$$

Proof 5.1-b uses '*Euler's Differential* Method' lemma 4, page 90.
The Continued Fraction we are trying to evaluate is converted into a differential equation which results after solving in a particular function and in turn this results in two particular integrals. (Which is the same integral as chosen a priori in proof 4.1-a.)
The quotient of two of these integrals finally brings the result of the Continued Fraction.

Proof 5.1-a

Consider the integral

$$I(n)=\int_0^{\frac{1}{4}\pi}\frac{(\cos(2x))^n}{\cos^2(x)}dx \qquad =\int_0^{\frac{1}{4}\pi}(\cos(2x))^n d\tan(x)$$

$$=\tan(x)\cdot(\cos(2x))^n\Big|_0^{\frac{1}{4}\pi}-\int_0^{\frac{1}{4}\pi}\tan(x)d\left(\cos(2x)\right)^n$$

$$2n\int_0^{\frac{1}{4}\pi}(\cos(2x))^{n-1}\sin(2x)\cdot\tan(x)dx=n\int_0^{\frac{1}{4}\pi}(\cos(2x))^{n-1}\sin(2x)\cdot\frac{2\sin(x)}{\cos(x)}dx$$

$$=n\int_0^{\frac{1}{4}\pi}(\cos(2x))^{n-1}\cdot\frac{\sin^2(2x)}{\cos^2(x)}dx$$

$$=n\int_0^{\frac{1}{4}\pi}\frac{(\cos(2x))^{n-1}}{\cos^2(x)}\cdot\left(1-\cos^2(2x)\right)dx$$

$$=n\int_0^{\frac{1}{4}\pi}\frac{(\cos(2x))^{n-1}}{\cos^2(x)}\cdot dx-n\int^{\frac{1}{4}\pi}\frac{(\cos(2x))^{n+1}}{\cos^2(x)}dx\Rightarrow$$

$$I(n)= \qquad =n\cdot I(n-1)-n\cdot I(n+1)$$

From the recursive equation $I(n)=n\cdot I(n-1)-n\cdot I(n+1)$ it follows that

$$\frac{I(n)}{I(n-1)}=\frac{1}{\frac{1}{n}+\frac{I(n+1)}{I(n)}}$$

Therefore, $\frac{I_1}{I_0}$ can be written as the Continued Fraction

$$\frac{I_1}{I_0}=\frac{1}{\frac{1}{1}+\frac{I_2}{I_1}}=\frac{1}{\frac{1}{1}+\frac{1}{\frac{1}{2}+\frac{I_3}{I_2}}}=\frac{1}{\frac{1}{1}+\frac{1}{\frac{1}{2}+\frac{1}{\frac{1}{3}+\frac{I_4}{I_3}}}}=\frac{1}{\frac{1}{1}+\frac{1}{\frac{1}{2}+\frac{1}{\frac{1}{3}+\frac{1}{\frac{1}{4}+\ddots}}}}$$

But $\frac{I_1}{I_0}$ can also be evaluated by

$$\begin{cases} I_1 = \int_0^{\frac{1}{4}\pi} \frac{\cos(2x)}{\cos^2(x)} dx = \int_0^{\frac{1}{4}\pi} \frac{2\cos^2(x)-1}{\cos^2(x)} dx = 2\Big|_0^{\frac{1}{4}\pi} - \tan(x)\Big|_0^{\frac{1}{4}\pi} = \frac{\pi}{2} - 1 \\ \\ I_0 = \int_0^{\frac{1}{4}\pi} \frac{1}{\cos^2(x)} dx = \tan(x)\Big|_0^{\frac{1}{4}\pi} = 1 \end{cases}$$

Therefore we have $\dfrac{\frac{\pi}{2}-1}{1} = \dfrac{I_1}{I_0} = \cfrac{1}{\frac{1}{1} + \cfrac{1}{\frac{1}{2} + \cfrac{1}{\frac{1}{3} + \cfrac{1}{\frac{1}{4} + \ddots}}}}$

After some small algebraic manipulations, theorem 5 follows easily.

Which proves that $\frac{1}{1} + \cfrac{1}{\frac{1}{2} + \cfrac{1}{\frac{1}{3} + \cfrac{1}{\frac{1}{4} + \cfrac{1}{\frac{1}{5} + \ddots}}}} = \dfrac{2}{\pi - 2}$

Proof 5.1-b

This proof applies '*Euler's Differential Method*', lemma 4, page 90.
With $a=1,\ \alpha=1,\ b=1,\ \beta=0,\ c=1,\ \gamma=1$

The integral expression $(1+n)\int_0^1 PR^n dx = \int_0^1 PR^{n+1}dx + (1+n)\int_0^1 PR^{n+2}dx$

Brings with $I_n = \int_0^1 PR^n dx$ the recursive expression $(1+n)I_n = I_{n+1} + (1+n)I_{n+2}$

Which can be converted as follows into a Continued Fraction:

$$w_n = \frac{I_{n+1}}{I_n} \Rightarrow \frac{(1+n)}{w_n} = 1+(1+n)w_{n+1} \Rightarrow w_n = \cfrac{1}{\frac{1}{1+n} + w_{n+1}} \Rightarrow w_0 = \cfrac{1}{\frac{1}{1} + \cfrac{1}{\frac{1}{2} + \cfrac{1}{\frac{1}{3} + \ddots}}}$$

But the integral expression $(1+n)\int_0^1 PR^n dx = \int_0^1 PR^{n+1}dx + (1+n)\int_0^1 PR^{n+2}dx$ can also be converted according to Euler into a system of differential equations.

$Pdx = \dfrac{SdR}{R^2-1}$ and $\dfrac{1}{S}dS = \dfrac{1}{R^2-1}dR$

By solving the second DV it follows that $\ln|S| = \frac{1}{2}\ln\left|\dfrac{1-R}{1+R}\right| \Rightarrow S = C\cdot\left(\dfrac{1-R}{1+R}\right)^{\frac{1}{2}}$

Euler method involves the requirement that $R^{n+1}S=0$ for $x=0$ and for $x=1$ and this requirement is satisfied with the choice $R(x)=x$

This is used in the evaluation of $\int_0^1 Pdx$ and $\int_0^1 PRdx$

The evaluation of $\int_0^1 Pdx$ is far from trivial.

It involves the substitution $x=\sin(\varphi$, integration by parts, taking the limit using l' Hôpital's rule.

$$\int_0^1 Pdx = -C\int_0^1 \frac{1}{\sqrt{1-x^2}\,(1+x)}dx = -C\int_0^{\frac{\pi}{2}} \frac{1}{(1+\sin(\varphi))}d\varphi = -C\int_0^{\frac{\pi}{2}} \frac{1-\sin(\varphi)}{\cos^2(\varphi)}d\varphi \quad =$$

$$\int_0^1 Pdx = -C\int_0^{\frac{\pi}{2}} 1-\sin(\varphi)d\tan(\varphi) = -C\cdot\lim_{\varphi\to\frac{\pi}{2}}\big(1-\sin(\varphi)\big)\tan(\varphi) + C\int_0^{\frac{\pi}{2}} \tan(\varphi)d1-\sin(\varphi) = -C$$

The integral $\int_0^1 PRdx = -C\int_0^1 \frac{x}{\sqrt{1-x^2}\,(1+x)}dx$ can be evaluated by splitting into parts.

$$\int_0^1 PRdx = -C\int_0^1 \frac{1+x}{\sqrt{1-x^2}\,(1+x)}dx + C\int_0^1 \frac{1}{\sqrt{1-x^2}\,(1+x)}dx$$

The first integral is easy to evaluate and the second part has already been evaluated.

$$\int_0^1 PRdx = -C\int_0^1 \frac{1+x}{\sqrt{1-x^2}\,(1+x)}dx + C\int_0^1 \frac{1}{\sqrt{1-x^2}\,(1+x)}dx = -C\arcsin(x)\Big|_0^{\frac{1}{2}\pi} + C = -\frac{1}{2}\pi C + C$$

In the quotient $\dfrac{\int_0^1 PRdx}{\int_0^1 Px}$ the factor C is cancelled out.

Therefore

$$w_0 = \frac{I_1}{I_0} = \frac{\int_0^1 PRdx}{\int_0^1 Pdx} = \frac{-\frac{1}{2}\pi C + C}{-C} = \frac{1}{2}\pi - 1 = \cfrac{1}{\frac{1}{1} + \cfrac{1}{\frac{1}{2} + \cfrac{1}{\frac{1}{3} + \ddots}}} \quad \Rightarrow \quad \frac{2}{\pi - 2} = \frac{1}{1} + \cfrac{1}{\frac{1}{2} + \cfrac{1}{\frac{1}{3} + \cfrac{1}{\frac{1}{4} + \ddots}}}$$

Since $I_n = \int_0^1 PR^n dx$ it is now clear which integral could have achieved directly the desired recursion property. (See page 33)

$$I_n = \int_0^1 PR^n dx = \int_0^1 \frac{x^n}{\sqrt{1-x^2}\,(1+x)}dx = \int_0^{\frac{\pi}{2}} \frac{\sin^n(\varphi)}{(1+\sin(\varphi))}d\varphi$$

Let nu $\varphi = \dfrac{\pi}{2} - 2\alpha$

It follows that $I_n = -2\int_{\frac{\pi}{4}}^0 \dfrac{\cos(2\alpha)^n}{(1+\cos(2\alpha))}d\alpha = \int_0^{\frac{\pi}{4}} \dfrac{\cos(2\alpha)^n}{\cos^2(\alpha)}d\alpha = \int_0^{\frac{\pi}{4}} \dfrac{\cos(2x)^n}{\cos^2(x)}dx$

Theorem 6

$$\frac{4}{\pi}=1+\cfrac{1^2}{3+\cfrac{2^2}{5+\cfrac{3^2}{7+\cfrac{4^2}{9+\ddots}}}}$$

This proof uses again '*Euler's Differential Method*' lemma 4, page 90.

From $w_n=\dfrac{n}{(1+2n)+(1+n)w_{n+1}} \Rightarrow w_1=\cfrac{1}{3+\cfrac{2^2}{5+\cfrac{3^2}{7+\ddots}}}$ follows

$nI_n=(1+2n)I_{n+1}+(1+n)I_{n+2}$

Now let $I_n=\int\limits_0^1 PR^n dx$

From lemma 4, page 90 met $a=0,\ \alpha=1,\ b=1,\ \beta=2,\ c=1,\ \gamma=1$ follows

$$w_1=\frac{I_2}{I_1}=\frac{\int\limits_0^1 PR^2dx}{\int\limits_0^1 PRdx}=\cfrac{1}{3+\cfrac{2^2}{5+\cfrac{3^2}{7+\ddots}}}$$

Also from lemma 4, page 90 follows $\dfrac{1}{S}dS=-\dfrac{1}{R}dR+\dfrac{R+1}{2R+R^2-1}dR$

Integration both sides results in:

$$\ln|S|=-\ln(R)+\tfrac{1}{2}\int\frac{1}{(R+1)^2-2}d\left(R+1\right)^2-2=-\ln(R)+\tfrac{1}{2}\ln\left|\left(R+1\right)^2-2)\right|$$

Now choose $R(x)=\left(\sqrt{2}-1\right)x$ because this particular choice satisfies the requirement that $R^{n+1}S=0$ for $x=0$ and $x=1$

Since $\left(R+1\right)^2-2<0$ for $x=0$ and $x=1$ we achieve

$$S=C\cdot\frac{1}{R}\cdot\sqrt{1-2R-R^2}=C\cdot\frac{1}{R}\cdot\sqrt{2-\left(R+1\right)^2}$$

Now from $Pdx=\dfrac{SdR}{\beta R+\gamma R^2-\alpha}=\dfrac{SdR}{2R+R^2-1}$ it follows that

$$PRdx=\frac{SRdR}{2R+R^2-1}=-\frac{C}{\sqrt{2-\left(R+1\right)^2}}$$

We are now able to evaluate $\int_0^1 PR\,dx$ and $\int_0^1 PR^2 dx$

$$w_1 = \frac{I_2}{I_1} = \frac{\int_0^1 PR^2 dx}{\int_0^1 PR dx} = \frac{-C\int_0^{\sqrt{2}-1} \frac{R}{\sqrt{2-(R+1)^2}} dR}{-C\int_0^{\sqrt{2}-1} \frac{1}{\sqrt{2-(R+1)^2}} dR} = \frac{Numerator}{Denumerator}$$

$$Numerator = \int_0^{\sqrt{2}-1} \frac{R}{\sqrt{2-(R+1)^2}} dR = \sqrt{2}\int_{\frac{1}{\sqrt{2}}}^1 \frac{x}{\sqrt{1-x^2}} dx - \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{\sqrt{1-x^2}} dx =$$

$$= -\sqrt{2}\left(1-x^2\right)^{\frac{1}{2}}\Big|_{\frac{1}{\sqrt{2}}}^1 - \arcsin(x)\Big|_{\frac{1}{\sqrt{2}}}^1 = 1 - \frac{\pi}{4}$$

$$Denumerator = \int_0^{\sqrt{2}-1} \frac{1}{\sqrt{2-(R+1)^2}} dR = \int_0^{\sqrt{2}-1} \frac{1}{\sqrt{1-\left(\frac{R+1}{\sqrt{2}}\right)^2}} d\frac{R+1}{\sqrt{2}} = \arcsin\left(\frac{R+1}{\sqrt{2}}\right)\Bigg|_0^{\sqrt{2}-1} = \frac{\pi}{4}$$

Which results finally in: $w_1 = \frac{\int_0^1 PR^2 dx}{\int_0^1 PR dx} = \frac{Numerator}{Denumerator} = \frac{1-\frac{\pi}{4}}{\frac{\pi}{4}} = \cfrac{1}{3+\cfrac{2^2}{5+\cfrac{3^2}{7+\cfrac{4^2}{9+\ddots}}}}$

And after a few small algebraic manipulations follows theorem 6

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{3+\cfrac{2^2}{5+\cfrac{3^2}{7+\cfrac{4^2}{9+\ddots}}}}$$

Theorem 7.1

$$0+\cfrac{1^2}{0+\cfrac{2^2}{0+\cfrac{3^2}{0+\cfrac{4^2}{0+\ddots}}}}=\frac{\pi}{2}$$

Theorem 7.2

$$2+\cfrac{1^2}{2+\cfrac{2^2}{2+\cfrac{3^2}{2+\cfrac{4^2}{2+\ddots}}}}=\frac{2}{4-\pi}$$

Theorem 7.3

$$4+\cfrac{1^2}{4+\cfrac{2^2}{4+\cfrac{3^2}{4+\cfrac{4^2}{4+\ddots}}}}=\frac{6}{3\pi-8}$$

Theorem 7.4

$$6+\cfrac{1^2}{6+\cfrac{2^2}{6+\cfrac{3^2}{6+\cfrac{4^2}{6+\ddots}}}}=\frac{30}{52-15\pi}$$

Theorem 7.5

$$8+\cfrac{1^2}{8+\cfrac{2^2}{8+\cfrac{3^2}{8+\cfrac{4^2}{8+\ddots}}}}=\frac{210}{105\pi-304}$$

These 5 theorems can be extended further and further because of the theorem

$$\frac{1}{s+\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{n^2}{s}\right)} = 2\int_0^1 \frac{x^s}{1+x^2}\,dx$$

Obviously, these Continued Fractions can be calculated with a not very difficult integral evaluation.

But the general formula $\dfrac{1}{s+\underset{n=1}{\overset{\infty}{\mathrm{K}}}(\frac{n^2}{s})} = 2\int_0^1 \dfrac{x^s}{1+x^2}dx$ is difficult to prove!

The proof given here (in lemma 5, page 92) is an adaptation of a proof from is S. Khruschev. It uses –again- '*Euler's Differential Method*' lemma 4, page 90

Having proved $\dfrac{1}{s+\underset{n=1}{\overset{\infty}{\mathrm{K}}}(\frac{n^2}{s})} = 2\int_0^1 \dfrac{x^s}{1+x^2}dx$ it follows with a simple evaluation that:

$$2+\underset{n=1}{\overset{\infty}{\mathrm{K}}}(\tfrac{n^2}{2}) = 2+\cfrac{1^2}{2+\cfrac{2^2}{2+\cfrac{3^2}{2+\cfrac{4^2}{2+\ddots}}}} = \frac{2}{4-\pi}$$

$$4+\underset{n=1}{\overset{\infty}{\mathrm{K}}}(\tfrac{n^2}{4}) = 4+\cfrac{1^2}{4+\cfrac{2^2}{4+\cfrac{3^2}{4+\cfrac{4^2}{4+\ddots}}}} = \frac{6}{3\pi-8}$$

$$6+\underset{n=1}{\overset{\infty}{\mathrm{K}}}(\tfrac{n^2}{6}) = 6+\cfrac{1^2}{6+\cfrac{2^2}{6+\cfrac{3^2}{6+\cfrac{4^2}{6+\ddots}}}} = \frac{30}{52-15\pi}$$

$$8+\underset{n=1}{\overset{\infty}{\mathrm{K}}}(\tfrac{n^2}{8}) = 8+\cfrac{1^2}{8+\cfrac{2^2}{8+\cfrac{3^2}{8+\cfrac{4^2}{8+\ddots}}}} = \frac{210}{105\pi-304}$$

(The result at $s=8$ is wrong in Khrushchev, S. (2008). *Orthogonal Polynomials and Continued Fractions: From Euler's Point of View*

A remarkable Continued Fraction is obtained by letting $s=0$

$$0+\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\tfrac{n^2}{0}\right)=0+\cfrac{1^2}{0+\cfrac{2^2}{0+\cfrac{3^2}{0+\cfrac{4^2}{0+\ddots}}}}=\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7^2}{8^2}\cdot....=\frac{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8}\cdot...=\frac{2}{\pi}$$

This infinite product is also known as the 'Wallis product'. (John Wallis, 1655)

$$\prod_{n=1}^{\infty}\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdot\frac{10}{9}\cdot\frac{10}{11}...=\frac{\pi}{2}$$

Another observation is that apparently:

$$\frac{1}{s+\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{n^2}{s}\right)}=2\int_0^1\frac{x^s}{1+x^2}dx \;=2\int_0^1 x^s\sum_{k=0}^{\infty}(-x^2)^k dx \;=2\sum_{k=0}^{\infty}(-1)^k\int_0^1 x^{s+2k}dx \;=2\sum_{k=0}^{\infty}\frac{(-1)^k}{s+2k+1}$$

Again by letting $s=0$ it follows that:

$$\frac{1}{s+\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{n^2}{s}\right)}=2\int_0^1\frac{x^s}{1+x^2}dx \;=2\int_0^1 x^s\sum_{k=0}^{\infty}(-x^2)^k dx \;=2\sum_{k=0}^{\infty}(-1)^k\int_0^1 x^{s+2k}dx \;=2\sum_{k=0}^{\infty}\frac{(-1)^k}{s+2k+1}$$

$$\frac{1}{0+\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{n^2}{0}\right)}=\frac{\pi}{2}=2\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}=2\cdot\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+..\right)$$

Which is the well-known series by Gregory-Leibniz $\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}...$

Theorem 8.1

$$2+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ddots}}}}=\frac{4}{\pi}+1$$

Theorem 8.2

$$6+\cfrac{1^2}{6+\cfrac{3^2}{6+\cfrac{5^2}{6+\cfrac{7^2}{6+\ddots}}}}=\pi+3$$

Theorem 8.3

$$10+\cfrac{1^2}{10+\cfrac{3^2}{10+\cfrac{5^2}{10+\cfrac{7^2}{10+\ddots}}}}=\frac{16}{\pi}+5$$

Theorem 8.4

$$14+\cfrac{1^2}{14+\cfrac{3^2}{14+\cfrac{5^2}{14+\cfrac{7^2}{14+\ddots}}}}=\frac{9}{4}\pi+7$$

Theorem 8.5

$$18+\cfrac{1^2}{18+\cfrac{3^2}{18+\cfrac{5^2}{18+\cfrac{7^2}{18+\ddots}}}}=\frac{256}{9\pi}+9$$

Theorem 8.6

$$22+\cfrac{1^2}{22+\cfrac{3^2}{22+\cfrac{5^2}{22+\cfrac{7^2}{22+\ddots}}}}=\frac{225}{64}\pi+11$$

These theorems are all applications of two more general theorems already known to Wallis and Euler.

Theorem 8.1, 8.3, 8.5:

$$4k+1+\cfrac{1^2}{2(4k+1)+\cfrac{3^2}{2(4k+1)+\cfrac{5^2}{2(4k+1)+\cfrac{7^2}{2(4k+1)+\ddots}}}} = (2k+1)\frac{1}{W(k)}\cdot\frac{4}{\pi}$$

$$\text{met } W(k)=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\ldots\frac{(2k-1)\cdot(2k+1)}{2k\cdot 2k}$$

Theorem 8.2, 8.4, 8.6:

$$4k+3+\cfrac{1^2}{2(4k+3)+\cfrac{3^2}{2(4k+3)+\cfrac{5^2}{2(4k+3)+\cfrac{7^2}{2(4k+3)+\ddots}}}} = (2k+1)W(k)\cdot\pi$$

$$\text{met } W(k)=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\ldots\frac{(2k-1)\cdot(2k+1)}{2k\cdot 2k}$$

These theorems were later proved more generally by Ramanujan and Stieltjes.
Here both theorems are proved separately. The proof of theorem 8.1, 8.3 and 8.5 uses "*Euler's Differential Method"* lemma 4, page 90, but clever manipulation is still needed to fully prove the theorem.
The proof of theorem 8.2, 8.4 and 8.6 follows along similar lines.

A proof of theorem 8.1, 8.3 and 8.5:

The recursion $(1+2n)I_n = 2\cdot(4k+1)I_{n+1} + (2n+3)I_{n+2}$ results with $w_n = \frac{I_{n+1}}{I_n}$ in the Continued Fraction:

$$w_0 = \frac{1+2n}{2(4k+1)+(2n+3)w_1} \Rightarrow w_0 = \cfrac{1^2}{2(4k+1)+\cfrac{3^2}{2(4k+1)+\cfrac{5^2}{2(4k+1)+\ddots}}}$$

Lemma 4, page 90, states:

$$Pdx = \frac{SdR}{\beta R+\gamma R^2-\alpha} \text{ and } \frac{1}{S}dS = \frac{(a-\alpha)}{\alpha R}dR + \frac{(\alpha b - a\beta)+(\alpha c - a\gamma)R}{\alpha\left(\beta R+\gamma R^2-\alpha\right)}dR$$

With $a=1,\ \alpha=2,\ b=2\cdot(4k+1),\ \beta=0,\ c=3,\ \gamma=2$ and $k=0,1,2,...$ it follows

$$\frac{1}{S}dS = -\tfrac{1}{2}\frac{1}{R}dR + \frac{(4k+1)+R}{(R^2-1)}dR \Rightarrow \ln|S| = -\tfrac{1}{2}\ln|R| + (2k+\tfrac{1}{2})\ln\left|\frac{1-R}{1+R}\right| + \tfrac{1}{2}\ln\left|R^2-1\right| \Rightarrow$$

$$S = C\cdot\frac{1}{R^{\frac{1}{2}}}\cdot\left(\frac{1-R}{1+R}\right)^{2k+\frac{1}{2}}\cdot\sqrt{1-R^2}$$

Now choose $R=x^2 \Rightarrow S = C\cdot\frac{1}{x}\cdot\left(\frac{1-x^2}{1+x^2}\right)^{2k+\frac{1}{2}}\cdot\sqrt{1-x^4}$

With $C=1$ this results in $I_{0_k} = \int_0^1 Pdx = \int_0^1 \frac{S}{2(R^2-1)}dR = \int_0^1 \frac{\left(1-x^2\right)^{2k}}{\left(1+x^2\right)^{2k+1}}dx$

This integral can be evaluated using the substitution $x := \tan(2x)$

After some algebraic manipulations we achieve $I_{0_k} = \int_0^1 Pdx = \frac{1}{2}\int_0^{\frac{1}{2}\pi}\cos^{2k}(x)dx$

This is a so-called "reduction integral": $I_{0_k} = \frac{1}{2}\int_0^{\frac{1}{2}\pi}\cos^{2k}(x)dx = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot....\frac{2k-1}{2k}\cdot\frac{\pi}{4}$

The pattern becomes easily visible by evaluating the integral for $k=1,\ 2,\ 3,\ 4$ etc.

$$I_{0_{k=1}} = \frac{1}{2}\cdot\frac{\pi}{4} \qquad I_{0_{k=2}} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\pi}{4} \qquad I_{0_{k=3}} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{\pi}{4} \qquad I_{0_{k=4}} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{\pi}{4} \text{ enz.} \Rightarrow$$

The second integral $I_{1_k}=\int_0^1 PRdx=\int_0^1 \frac{RS}{2(R^2-1)}dR=C\int_0^1 \frac{x^2\left(1-x^2\right)^{2k}}{\left(1+x^2\right)^{2k+1}}dx$ can be also evaluated by some algebraic manipulations and the substitution $x:=\tan(2x)$

$$I_{1_k}=2k\int_0^{\frac{1}{2}\pi}\cos^{2k-1}(x)dx-(4k+1)I_0$$

The first term is again a reduction integral. $I_{1_k}=1\cdot\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2k}{2k-1}-(4k+1)I_{0_k}$

This allows to evaluate: $w_0=\frac{I_1}{I_0}=\frac{2k\int_0^{\frac{1}{2}\pi}\cos^{2k-1}(x)dx-(4k+1)I_0}{I_0}\Rightarrow$

$$w_0+(4k+1)=(4k+1)+\cfrac{1^2}{2(4k+1)+\cfrac{3^2}{2(4k+1)+\ddots}}=\frac{2k\int_0^{\frac{1}{2}\pi}\cos^{2k-1}(x)dx}{I_0}\Rightarrow$$

$$(4k+1)+\cfrac{1^2}{2(4k+1)+\cfrac{3^2}{2(4k+1)+\ddots}}=\frac{1\cdot\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2k}{2k-1}}{\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\cdot\frac{2k-1}{2k}}\cdot\frac{4}{\pi}=$$

$$=\frac{(2k)\cdot(2k)}{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\ldots.(2k-1)(2k-1)}\cdot\frac{4}{\pi}$$

With a small trick we bring the expression of the denominator in the form of the Wallis product. We multiply, divide and shift some terms. For example with 7

$$\frac{7\cdot(2k)\cdot(2k)}{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot 7\ldots.(2k-1)(2k-1)}\cdot\frac{4}{\pi}\Rightarrow\frac{(2k+1)\cdot(2k)\cdot(2k)}{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\ldots(2k-1)(2k+1)}\cdot\frac{4}{\pi}$$

Finally it follows that

$$(4k+1)+\cfrac{1^2}{2(4k+1)+\cfrac{3^2}{2(4k+1)+\ddots}}=(2k+1)\frac{1}{W(k)}\cdot\frac{4}{\pi}$$

$$\text{with } W(k)=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdots\frac{(2k-1)\cdot(2k+1)}{2k\cdot 2k}$$

A proof of theorem 8.2, 8.4 and 8.6:

Also this theorem follows from '*Euler's Differential Method*' lemma 4, page 90.

The recursion $(1+2n)I_n = 2\cdot(4k+3)I_{n+1} + (2n+3)I_{n+2}$ results with $w_n = \dfrac{I_{n+1}}{I_n}$ in the Continued Fraction

$$w_0 = \frac{1+2n}{2(4k+3)+(2n+3)w_1} \Rightarrow w_0 = \cfrac{1^2}{2(4k+3)+\cfrac{3^2}{2(4k+3)+\cfrac{5^2}{2(4k+3)+\ddots}}}$$

Lemma 4, page 90, states:

$$Pdx = \frac{SdR}{\beta R + \gamma R^2 - \alpha} \text{ and } \frac{1}{S}dS = \frac{(a-\alpha)}{\alpha R}dR + \frac{(\alpha b - a\beta) + (\alpha c - a\gamma)R}{\alpha\left(\beta R + \gamma R^2 - \alpha\right)}dR$$

With $a=1,\ \alpha=2,\ b=2(4k+3),\ \beta=0,\ c=3,\ \gamma=2$ and $k=0,1,2,\ldots$ it follows

$$\frac{1}{S}dS = -\tfrac{1}{2}\frac{1}{R}dR + \frac{(4k+3)+R}{(R^2-1)}dR \Rightarrow \ln|S| = -\tfrac{1}{2}\ln|R| + (2k+\tfrac{3}{2})\ln\left|\frac{1-R}{1+R}\right| + \tfrac{1}{2}\ln\left|R^2-1\right| \Rightarrow$$

$$S = C\cdot\frac{1}{R^{\frac{1}{2}}}\cdot\left(\frac{1-R}{1+R}\right)^{2k+\frac{3}{2}}\cdot\sqrt{1-R^2}$$

Now choose $R = x^2 \Rightarrow S = C\cdot\dfrac{1}{x}\cdot\left(\dfrac{1-x^2}{1+x^2}\right)^{2k+\frac{3}{2}}\sqrt{1-x^4}$

which results in

$$I_{0_k} = \int_0^1 Pdx = \int_0^1 \frac{S}{2(R^2-1)}dR = C\cdot\int_0^1 \frac{\left(\frac{1-x^2}{1+x^2}\right)^{2k+\frac{3}{2}}}{\sqrt{1-x^4}}dx = C\cdot\int_0^1 \frac{\left(1-x^2\right)^{2k+1}}{\left(1+x^2\right)^{2k+2}}dx$$

Again, this integral can be evaluated using the substitution $x := \tan(2x)$

After some algebraic manipulations we achieve $I_{0_k} = \int_0^1 Pdx = \frac{1}{2}C\cdot\int_0^{\frac{\pi}{2}}\cos(x)^{2k+1}dx$

Again, this is a reduction integral.

With $C=1$ we evaluate $I_{0_k} = \dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{2}}\cos(x)^{2k+1}dx = k\cdot\frac{1}{1}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\ldots\cdot\frac{2k-2}{2k+1}$

(This can be done analytically but it is easier to spot the pattern by evaluating the integral for $k=1,\ 2,\ 3,\ 4$ etc.)

$$I_{0_{k=1}} = 1\cdot\frac{1}{3} \qquad I_{0_{k=2}} = 2\cdot\frac{1\cdot 2}{3\cdot 5} \qquad I_{0_{k=3}} = 3\cdot\frac{1\cdot 2\cdot 4}{3\cdot 5\cdot 7} \qquad I_{0_{k=4}} = 4\cdot\frac{1\cdot 2\cdot 4\cdot 6}{3\cdot 5\cdot 7\cdot 9} \text{ enz.} \Rightarrow$$

The second integral $I_{1_k} = \int_0^1 PRdx = \int_0^1 \frac{RS}{2(R^2-1)}dR = C\cdot\int_0^1 \frac{x^2\left(1-x^2\right)^{2k+1}}{\left(1+x^2\right)^{2k+2}}dx$ can be also evaluated by some algebraic manipulations and the substitution $x := \tan(2x)$

$$I_{1_k} = 2k\int_0^{\frac{1}{2}\pi} \cos^{2k-1}(x)dx - (4k+3)I_0$$

The first term is again a reduction integral.

$$I_{1_{k=1}} = \frac{1}{2}\cdot\frac{3}{2}\pi - \frac{7}{3} \qquad I_{1_{k=2}} = \frac{1}{2}\cdot\frac{3\cdot 5}{2\cdot 4}\pi - \frac{44}{15} \qquad I_{1_{k=3}} = \frac{1}{2}\cdot\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\pi - \frac{24}{7}$$

$$I_{1_{k=4}} = \frac{1}{2}\cdot\frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}\pi - \frac{1216}{315} \text{ etc.}$$

Therefore $I_{1_k} = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\ldots\frac{2k-1}{2k}\cdot\pi - (4k+3)\cdot I_{0_k}$

We apply again a small manipulation. Multiplying by $(2k+1)$, shifting everything in the numerator one position to the right and dividing by the same term (here is 9 used as an example) results in the reversed Wallis product :

$$\frac{I_{1_k}}{I_{0_k}} = \frac{\frac{1}{2}\cdot\frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}}{4\cdot\frac{1\cdot 2\cdot 4\cdot 6}{3\cdot 5\cdot 7\cdot 9}}\ldots = \frac{\frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}}{\frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9}}\ldots = \frac{3\cdot 3}{2\cdot 2}\cdot\frac{5\cdot 5}{4\cdot 4}\cdot\frac{7\cdot 7}{6\cdot 6}\cdot\frac{9\cdot 9}{8\cdot 8}\cdot\ldots\ldots =$$

$$= 9\cdot\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\ldots\frac{(2k-1)(2k+1)}{(2k)\cdot(2k)} \Rightarrow \frac{I_{1_k}}{I_{0_k}} = (2k+1)\cdot W(k)\cdot\pi$$

Which proves that:

$$(4k+3) + \cfrac{1^2}{2(4k+3) + \cfrac{3^2}{2(4k+3) + \ddots}} = (2k+1)W(k)\cdot\pi$$

$$\text{with } W(k) = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\ldots\frac{(2k-1)\cdot(2k+1)}{2k\cdot 2k}$$

Some Continued Fractions for *e*

The number *e* can be defined as the number satisfying the requirement that the derivative of $f(x) = e^x$ equals the function itself.
From this definition it is easy to show that *e* can also be defined as a limit

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Using the Binomium of Newton it is easy to derive the series expansion

$e = \sum_{n=0}^{\infty}\frac{1}{n!} = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots$ and also $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = \frac{1}{0!}+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$

Besides these classic definitions for *e*, several beautiful Continued Fractions exist.

Theorem 9.1

$$1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\ddots}}}} = e-1$$

Theorem 9.2

$$1+\cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{2}}{1+\cfrac{\frac{1}{3}}{1+\cfrac{\frac{1}{4}}{1+\ddots}}}} = e-1$$

Theorem 9.3

$$1+\cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{7+\cfrac{8}{9+\cfrac{10}{11+\ddots}}}}} = \frac{1}{\sqrt{e}-1}$$

Theorem 9.4

$$1+\cfrac{6}{4+\cfrac{12}{7+\cfrac{18}{10+\cfrac{24}{13+\ddots}}}} = \frac{2}{\sqrt[3]{e^2}-1}$$

In the proof of these 4 theorems, Euler's Continued Fraction formula is used and applied to the series expansion for e^x .

Proof:

From $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + ... = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ and using

$$a_0 + a_0a_1 + a_0a_1a_2 + ... + a_0a_1a_2....a_n = \cfrac{a_0}{1 - \cfrac{a_1}{1 + a_1 - \cfrac{a_2}{1 + a_2 - \cfrac{a_3}{1 + a_3 - \cfrac{a_4}{1 + a_4 - \ddots}}}}}$$

we conclude that $a_0 = 1,\ a_1 = x, a_2 = \frac{1}{2}x, a_3 = \frac{1}{3}x, a_4 = \frac{1}{4}x, a_5 = \frac{1}{5}x$ etc. $\Rightarrow$

$$e^x = \cfrac{1}{1 - \cfrac{x}{1 + x - \cfrac{\frac{1}{2}x}{1 + \frac{1}{2}x - \cfrac{\frac{1}{3}x}{1 + \frac{1}{3}x - \cfrac{\frac{1}{4}x}{1 + \frac{1}{4}x - \cfrac{\frac{1}{5}x}{1 + \frac{1}{5}x - \ddots}}}}}} \Rightarrow$$

With $x := -x$ it follows that $e^x = 1 + \cfrac{x}{1 - x + \cfrac{\frac{1}{2}x}{1 - \frac{1}{2}x + \cfrac{\frac{1}{3}x}{1 - \frac{1}{3}x + \cfrac{\frac{1}{4}x}{1 - \frac{1}{4}x + \cfrac{\frac{1}{5}x}{1 - \frac{1}{5}x + \ddots}}}}}$

Choosing $x = 1$ results in: $e = 1 + \cfrac{1}{\cfrac{\frac{1}{2}}{\frac{1}{2} + \cfrac{\frac{1}{3}}{\frac{2}{3} + \cfrac{\frac{1}{4}}{\frac{3}{4} + \cfrac{\frac{1}{5}}{\frac{4}{5} + \ddots}}}}}$ and with some small algebraic manipulations this can be written as theorem 9.1.

$$e - 1 = 1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \ddots}}}}$$

Applying an 'equivalence transformation' converts this theorem to Theorem 9.2:

$$e-1=1+\cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{2}}{1+\cfrac{\frac{1}{3}}{1+\cfrac{\frac{1}{4}}{1+\cfrac{\frac{1}{5}}{1+\ddots}}}}}$$

Choosing $x=\frac{1}{2}$ produces another Continued Fraction. $\sqrt{e}=1+\cfrac{\frac{1}{2}}{\frac{1}{2}+\cfrac{\frac{1}{2}}{\frac{3}{2}+\cfrac{1}{\frac{5}{2}+\cfrac{\frac{3}{2}}{\frac{7}{2}+\cfrac{2}{\frac{9}{2}+\ddots}}}}}$

Again, after minor manipulations theorem 9.3 follows.

$$\frac{1}{\sqrt{e}-1}=1+\cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{7+\cfrac{8}{9+\ddots}}}}$$

Choosing $x=\frac{2}{3}$ results in the Continued Fraction $\frac{2}{\sqrt[3]{e^2}-1}=1+\cfrac{2}{\frac{4}{3}+\cfrac{\frac{4}{3}}{\frac{7}{3}+\cfrac{2}{\frac{10}{3}+\cfrac{\frac{8}{3}}{\frac{13}{3}+\ddots}}}}$

and again after some manipulations follows theorem 9.4

$$\frac{2}{\sqrt[3]{e^2}-1}=1+\cfrac{6}{4+\cfrac{12}{7+\cfrac{18}{10+\cfrac{24}{13+\ddots}}}}$$

Theorem 10

$$1+\cfrac{1+\cfrac{1+\cfrac{1+\cfrac{1+\cfrac{1+\cfrac{1+\cfrac{1+\cdot^{\cdot^{\cdot}}}{7}}{6}}{5}}{4}}{3}}{2}}{1}=e$$

This is an astonishing beautiful expression. It is the ascending variant of a Continued Fraction". The proof consists of a demonstration that every additional term results in precisely the same approximation for e as resulting from its series expansion.

Proof:

From $e=1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{65!}+\frac{1}{7!}+\frac{1}{8!}+.....$ follows a series of successive rational approximations: $e=1,\ 1,\ \frac{5}{2},\frac{8}{3},\frac{63}{24},\frac{163}{60},\frac{1957}{720},\frac{685}{252},...$ etc.

The ascending Continued Fraction $e=1+\cfrac{1+\cfrac{1+\cfrac{1+\cfrac{1+\cfrac{1+\cfrac{1+\cfrac{1+\cdot^{\cdot^{\cdot}}}{7}}{6}}{5}}{4}}{3}}{2}}{1}$ produces precisely these same approximations as in $e=1,\ 1,\ \frac{5}{2},\frac{8}{3},\frac{63}{24},\frac{163}{60},\frac{1957}{720},\frac{685}{252},...$

Theorem 11.1

$$2+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\ddots}}}}=\frac{e+1}{e-1}$$

Explanation 11.1:

Any break e.g. $\frac{27818}{10000}$ can be numerically transformed into a Continued Fraction by a simple iterative process of inversion, division and splitting into the integer part and the fraction part.

For example:

$$\frac{271828}{100000}=2+\frac{71828}{100000}=1+\cfrac{1}{\cfrac{100000}{71828}}=1+\cfrac{1}{1+\cfrac{28172}{71828}}=1+\cfrac{1}{1+\cfrac{1}{\cfrac{71828}{28172}}}=$$

$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{15484}{28172}}}=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{\cfrac{28172}{15484}}}}=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{\cfrac{15484}{12688}}}}}=$$

$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{12688}{2796}}}}}}=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{\cfrac{2796}{15052}}}}}}}\ \text{enz.}$$

Applying that to the number $e = 2.71828182845904523536028 74...$ reveals a regular pattern:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \ddots}}}}}}}}}$$

Lemma 6, page 94 can be applied to remove those repetitive ones.
Afterwards, a small series of algebraic operations is needed to get the pattern regular.

With $A = 1,\ \ a = 2,\ \ b = 4,\ \ c = 6 \Rightarrow$

$$\frac{1}{e-2} = 1 + \cfrac{2}{5 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \ddots}}}} \Rightarrow e = 2 + \cfrac{1}{1 + \cfrac{2}{5 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \ddots}}}}} \Rightarrow$$

$$\frac{e+1}{e-1} = 2 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \ddots}}}}$$

But nothing has been proven yet….

Theorem 11.2

$$4+\cfrac{1}{12+\cfrac{1}{20+\cfrac{1}{28+\cfrac{1}{36+\ddots}}}}=\frac{\sqrt{e}+1}{\sqrt{e}-1}$$

Explanation 11.2:

In the same way, it is also possible to convert $\sqrt{e}$ to a Continued Fraction and again a simple pattern appears to occur.

$$\sqrt{e}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{9+\ddots}}}}}}}$$

Again lemma 6, page 94 can be applied to remove the repeated ones. Afterwards, a small series of algebraic operations is needed to get the pattern regular.

With $A=1,\ a=1\ \ b=5,\ c=9\Rightarrow$

$$\sqrt{e}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{9+\ddots}}}}}}}=1+\cfrac{2}{3+\cfrac{1}{12+\cfrac{1}{20+\cfrac{1}{28+\ddots}}}}\Rightarrow$$

$$\frac{\sqrt{e}+1}{\sqrt{e}-1}=4+\cfrac{1}{12+\cfrac{1}{20+\cfrac{1}{28+\ddots}}}$$

But still nothing has been proven….

Theorem 11.3

$$6+\cfrac{1}{18+\cfrac{1}{30+\cfrac{1}{42+\cfrac{1}{54+\ddots}}}}=\frac{\sqrt[3]{e}-1}{\sqrt[3]{e}-1}$$

Explanation 11.3:

In the same way, it is also possible to convert $\sqrt[3]{e}$ to a Continued Fraction and again a simple and regular pattern appears to occur.

$$\sqrt[3]{e}=1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{8+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{14+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{20+\ddots}}}}}}}}}}$$

Again lemma 6, page 94 can be applied to remove the repeated ones. Afterwards, a small series of algebraic operations is needed to get the pattern regular.

With $A=1,\ \ a=2\ \ \ b=8,\ \ c=14\Rightarrow$

$$\sqrt[3]{e}=1+\cfrac{2}{5+\cfrac{1}{18+\cfrac{1}{30+\cfrac{1}{42+\cfrac{1}{54+\ddots}}}}}\ \Rightarrow$$

$$\frac{\sqrt[3]{e}-1}{\sqrt[3]{e}-1}=6+\cfrac{1}{18+\cfrac{1}{30+\cfrac{1}{42+\cfrac{1}{54+\ddots}}}}$$

But this also proves nothing…

Theorem 11.4

$$8+\cfrac{1}{24+\cfrac{1}{40+\cfrac{1}{42+\cfrac{1}{56+\ddots}}}}=\frac{\sqrt[4]{e}-1}{\sqrt[4]{e}-1}$$

Explanation 11.4:

In the same way, it is also possible to convert $\sqrt[4]{e}$ to a Continued Fraction and again a simple and regular pattern appears to occur.

$$\sqrt[4]{e}=1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{11+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{19+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{27+\ddots}}}}}}}}}}$$

Again lemma 6, page 95 can be applied to remove the repeated ones. Afterwards, a small series of algebraic operations is needed to get the pattern regular.

With $A=1,\ \ a=3\ \ b=11,\ \ c=19 \Rightarrow$

$$\sqrt[4]{e}=1+\cfrac{2}{7+\cfrac{1}{24+\cfrac{1}{40+\cfrac{1}{56+\cfrac{1}{72+\ddots}}}}}\quad\Rightarrow$$

$$\frac{\sqrt[4]{e}+1}{\sqrt[4]{e}-1}=8+\cfrac{1}{24+\cfrac{1}{40+\cfrac{1}{56+\cfrac{1}{72+\ddots}}}}$$

It is likely that this pattern will continue, but nothing has been proven yet….

Theorem 11.1 to 11.4 show a remarkable pattern.

$$\frac{e+1}{e-1}=2+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\ddots}}} \qquad \frac{\sqrt{e}+1}{\sqrt{e}-1}=4+\cfrac{1}{12+\cfrac{1}{20+\cfrac{1}{28+\ddots}}}$$

$$\frac{\sqrt[3]{e}-1}{\sqrt[3]{e}-1}=6+\cfrac{1}{18+\cfrac{1}{30+\cfrac{1}{42+\ddots}}} \qquad \frac{\sqrt[4]{e}+1}{\sqrt[4]{e}-1}=8+\cfrac{1}{24+\cfrac{1}{40+\cfrac{1}{56+\ddots}}}$$

In generalized form:

$$\frac{e^{\frac{1}{x}}+1}{e^{\frac{1}{x}}-1}=2x+\cfrac{1}{6x+\cfrac{1}{10x+\cfrac{1}{14x+\cfrac{1}{18x+\ddots}}}} \text{ of ook } \frac{e^{\frac{2}{x}}+1}{e^{\frac{2}{x}}-1}=x+\cfrac{1}{3x+\cfrac{1}{5x+\cfrac{1}{7x+\cfrac{1}{9x+\ddots}}}}$$

By moving to a new variable it follows: $$\frac{e^{2x}+1}{e^{2x}-1}=\tfrac{1}{x}+\cfrac{1}{\tfrac{3}{x}+\cfrac{1}{\tfrac{5}{x}+\cfrac{1}{\tfrac{7}{x}+\cfrac{1}{\tfrac{9}{x}+\ddots}}}}$$

Because $\sinh(x)=\dfrac{e^{x}-e^{-x}}{2}$ and $\cosh(x)=\dfrac{e^{x}+e^{-x}}{2}$, $\coth(x)=\dfrac{e^{x}+e^{-x}}{e^{x}-e^{-x}}=\dfrac{e^{2x}+1}{e^{2x}-1}$

Thus, the observed pattern for the Continued fraction allows the assumption that

$$\coth(x)=\tfrac{1}{x}+\cfrac{1}{\tfrac{3}{x}+\cfrac{1}{\tfrac{5}{x}+\cfrac{1}{\tfrac{7}{x}+\cfrac{1}{\tfrac{9}{x}+\ddots}}}}$$

Conversely, if it could be proven that $\coth(x)$ satisfies this expression, the correctness of theorems 11.1 to 11.4 directly follows.
It is therefore to prove that:

$$\coth(x)=\tfrac{1}{x}+\cfrac{1}{\tfrac{3}{x}+\cfrac{1}{\tfrac{5}{x}+\cfrac{1}{\tfrac{7}{x}+\cfrac{1}{\tfrac{9}{x}+\ddots}}}}$$ which implies $$1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\ddots}}}}=\frac{e^{2}+1}{e^{2}-1}$$

Theorem 12

$$1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\ddots}}}}}=\frac{e^2+1}{e^2-1}$$

Proof:

Euler proved that this continued fraction is the solution of the *´Riccati´* differential equation, whose solution is also $\coth(x)$.

But his proof is not clear (to me).

The theorem to prove is: $\coth(x)=\dfrac{e^{2x}+1}{e^{2x}-1}=\frac{1}{x}+\cfrac{1}{\frac{3}{x}+\cfrac{1}{\frac{5}{x}+\cfrac{1}{\frac{7}{x}+\cfrac{1}{\frac{9}{x}+\ddots}}}}$

If this is proven, choosing *x=1* results immediately in theorem 12.

Gauss proved the theorem by using the HyperGeometric series.

Consider the HyperGeometric series

$$F(a,x)=1+\frac{1}{a}\cdot\frac{x^1}{1!}+\frac{1}{a\cdot(a+1)}\cdot\frac{x^2}{2!}+\frac{1}{a(a+1)(a+2)}\cdot\frac{x^3}{3!}+....$$

Consequently,

$$F(a+1,x)=1+\frac{1}{a+1}\cdot\frac{x^1}{1!}+\frac{1}{(a+1)\cdot(a+2)}\cdot\frac{x^2}{2!}+\frac{1}{(a+1)(a+2)(a+3)}\cdot\frac{x^3}{3!}+....$$

There appears to be a recursive relationship between $F(a,x)$, $F(a+1,x)$ and $F(a+2,x)$.

They satisfy $F(a,x)-F(a+1,x)=\dfrac{x}{a(a+1)}\cdot F(a+2,x)$

Proof:

$$F(a,x)-F(a+1,x)=\left(\frac{1}{a}-\frac{1}{a+1}\right)\frac{x^1}{1!}+\left(\frac{1}{a\cdot(a+1)}-\frac{1}{(a+1)\cdot(a+2)}\right)\frac{x^2}{2!}+$$

$$+\left(\frac{1}{a(a+1)(a+2)}-\frac{1}{(a+1)(a+2)(a+3)}\right)\frac{x^3}{3!}+..$$

$$F(a,x)-F(a+1,x)=\frac{1}{a+1}\cdot\frac{x^1}{1!}+\frac{2}{a\cdot(a+1)(a+2)}\cdot\frac{x^2}{2!}+\frac{3}{a(a+1)(a+2)(a+3)}\cdot\frac{x^3}{3!}+\ldots\Rightarrow$$

$$F(a,x)-F(a+1,x)=\frac{x}{a(a+1)}\cdot\left(1+\frac{1}{(a+2)}\cdot\frac{x^1}{1!}+\frac{1}{(a+2)(a+3)}\cdot\frac{x^2}{2!}+.....\right)\Rightarrow$$

$$F(a,x)-F(a+1,x)=\frac{x}{a(a+1)}\cdot F(a+2,x)$$

From the recursion it follows:

$$\frac{F(a,x)}{F(a+1,x)}=1+\frac{x}{a(a+1)}\cdot\frac{F(a+2,x)}{F(a+1,x)}\Rightarrow\frac{F(a+1,x)}{F(a,x)}=\frac{1}{1+\frac{x}{a(a+1)}\cdot\frac{F(a+2,x)}{F(a+1,x)}}$$

And from this again follows in turn the Continued Fraction:

$$\frac{F(a+1,x)}{F(a,x)}=\frac{1}{1+\frac{x}{a(a+1)}\cdot\frac{F(a+2,x)}{F(a+1,x)}}=\frac{1}{1+\frac{\frac{x}{a(a+1)}}{1+\frac{F(a+3,x)}{F(a+2,x)}}}=\frac{1}{1+\cfrac{\frac{x}{a(a+1)}}{1+\cfrac{\frac{x}{(a+1)(a+2)})}{1+\cfrac{\frac{x}{(a+2)(a+3)})}{1+\cfrac{\frac{x}{(a+3)(a+4)})}{1+\ddots}}}}}$$

Now choose $a=\frac{1}{2}$ and $x:=\frac{x^2}{4}$

This results in $\frac{F(\frac{3}{2},\frac{x^2}{4})}{F(\frac{1}{2},\frac{x^2}{4})}=\cfrac{1}{1+\cfrac{\frac{x^2}{1\cdot3}}{1+\cfrac{\frac{x^2}{3\cdot5}}{1+\cfrac{\frac{x^2}{5\cdot7}}{1+\cfrac{\frac{x^2}{7\cdot9}}{1+\ddots}}}}}=\cfrac{1}{1+\cfrac{x^2}{3+\cfrac{x^2}{5+\cfrac{x^2}{7+\cfrac{x^2}{9+\ddots}}}}}$

So we have found an expression for the Continued Fraction.

Now we evaluate this expression in a different way.

The Taylor series expansion for $\cosh(x)$ is: $\cosh(x)=1+\frac{x^2}{2}+\frac{x^4}{4!}+\frac{x^6}{6!}+\frac{x^8}{8!}....$

But this equals $F(\frac{1}{2},\frac{x^2}{4})$ because

$$F(\tfrac{1}{2},\tfrac{x^2}{4})=1+2\cdot\tfrac{x^2}{4}+\frac{2^2}{1\cdot3}\cdot\frac{x^4}{4^2\cdot2!}+\frac{2^3}{1\cdot3\cdot5}\cdot\frac{x^6}{4^3\cdot3!}+\frac{2^4}{1\cdot3\cdot5\cdot7}\cdot\frac{x^8}{4^4\cdot4!}...\Rightarrow$$

So $F(\frac{1}{2},\frac{x^2}{4})=1+\frac{x^2}{2}+\frac{x^4}{4!}+\frac{x^6}{6!}+\frac{x^8}{8!}...=\cosh(x)$

The Taylor series expansion for $\sinh(x)$ is: $\sinh(x)=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+....$

This equals $F(\frac{3}{2},\frac{x^2}{4})$ because

$$F(\tfrac{3}{2},\tfrac{x^2}{4})=1+\frac{2}{3}\cdot\frac{x^2}{4\cdot1!}+\frac{2^2}{3\cdot5}\cdot\frac{x^4}{4^2\cdot2!}+\frac{2^3}{3\cdot5\cdot7}\cdot\frac{x^6}{4^3\cdot3!}+....$$

So $x\cdot F(\frac{3}{2},\frac{x^2}{4})=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+....=\sinh(x)$

Combined, it follows that:

$$\frac{F(\frac{3}{2},\frac{x^2}{4})}{F(\frac{1}{2},\frac{x^2}{4})}=\frac{1}{x}\cdot\frac{\sinh(x)}{\cosh(x)}=\cfrac{1}{1+\cfrac{x^2}{3+\cfrac{x^2}{5+\cfrac{x^2}{7+\ddots}}}}\Rightarrow\tanh(x)=\frac{\sinh(x)}{\cosh(x)}=\cfrac{x}{1+\cfrac{x^2}{3+\cfrac{x^2}{5+\cfrac{x^2}{7+\ddots}}}}\Rightarrow$$

This can be converted to $\tanh(x)=\cfrac{1}{\frac{1}{x}+\cfrac{1}{\frac{3}{x}+\cfrac{1}{\frac{5}{x}+\cfrac{1}{\frac{7}{x}+\ddots}}}}$ and therefore

$$\coth(x)=\frac{e^{2x}+1}{e^{2x}-1}=\frac{1}{x}+\cfrac{1}{\frac{3}{x}+\cfrac{1}{\frac{5}{x}+\cfrac{1}{\frac{7}{x}+\cfrac{1}{\frac{9}{x}+\ddots}}}}$$

This proves the theorems 11.1 to 11.4.

By choosing $x=1$, the beautiful Continued Fraction of theorem 12.1 results.

$$\frac{e^2+1}{e^2-1}=1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\ddots}}}}}$$

Theorem 13.1

$$\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\ddots}}}}=\frac{1}{e-1}$$

Theorem 13.2

$$1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{5+\ddots}}}}=\frac{1}{e-2}$$

Theorem 13.3

$$\cfrac{2}{1+\cfrac{4}{2+\cfrac{6}{3+\cfrac{8}{4+\ddots}}}}=\frac{4}{e^2-3}$$

Theorem 13.4

$$1+\cfrac{2}{2+\cfrac{4}{3+\cfrac{6}{4+\cfrac{8}{5+\ddots}}}}=\frac{4}{e^2-5}$$

These Continued Fractions follow immediately from the powerful lemma 7, page 95:

$$s+\mathrm{K}\left(\frac{pn}{s+n}\right)=\frac{1}{e^p\int_0^1 x^{s+p-1}e^{-px}dx}$$

For $s=0$ and $p=1 \Rightarrow$ $$\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\ddots}}}}}=\frac{1}{e\int_0^1 e^{-x}dx}=\frac{1}{e-1}$$

For $s=1$ and $p=1 \Rightarrow$ $$1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{5+\ddots}}}}=\frac{1}{e\int_0^1 xe^{-x}dx}=\frac{1}{e-2}$$

For $s=0$ and $p=2 \Rightarrow$ $$\cfrac{2}{1+\cfrac{4}{2+\cfrac{6}{3+\cfrac{8}{4+\cfrac{10}{5+\ddots}}}}}=\frac{1}{e^2\int_0^1 xe^{-2x}dx}=\frac{4}{e^2-3}$$

For $s=1$ and $p=2 \Rightarrow$ $$1+\cfrac{2}{2+\cfrac{4}{3+\cfrac{6}{4+\cfrac{8}{5+\ddots}}}}=\frac{1}{e^2\int_0^1 x^2e^{-2x}dx}=\frac{4}{e^2-5}$$

Some Continued Fractions resulting in integral expressions

Theorem 14

$$1+\cfrac{1}{1+\cfrac{3}{1+\cfrac{5}{1+\cfrac{7}{1+\cfrac{9}{1+\ddots}}}}}=1+\frac{2\int\limits_0^\infty x^{\frac{1}{2}}e^{-x-x^2}dx}{\int\limits_0^\infty x^{-\frac{1}{2}}e^{-x-x^2}dx}$$

Proof:

In this Continued Fraction a recursive expression the following recursion is hidden.
$I_n=-2nI_n+2I_{n+1}+4I_{n+2}$

This is easy to see when rewriting as follows:
$$(2n+1)I_n=2I_{n+1}+4I_{n+2}\Rightarrow(2n+1)\frac{I_n}{I_{n+1}}=2+4\frac{I_{n+2}}{I_{n+1}}\Rightarrow(n+\tfrac{1}{2})\frac{I_n}{I_{n+1}}=1+2\frac{I_{n+2}}{I_{n+1}}$$

Now let $w_n=\dfrac{I_{n+1}}{I_n}$ From this it follows that $\dfrac{(n+\frac{1}{2})}{w_n}=1+2w_{n+1}\quad\Rightarrow w_n=\dfrac{(n+\frac{1}{2})}{1+2w_{n+1}}$
And consequently
$$w_0=\frac{\frac{1}{2}}{1+2w_1}=\cfrac{\frac{1}{2}}{1+2\cdot\cfrac{\frac{3}{2}}{1+2\cdot w_2}}=\cfrac{\frac{1}{2}}{1+2\cdot\cfrac{\frac{3}{2}}{1+2\cdot\cfrac{\frac{5}{2}}{1+2\cdot w_3}}}=\cfrac{\frac{1}{2}}{1+2\cdot\cfrac{\frac{3}{2}}{1+2\cdot\cfrac{\frac{5}{2}}{1+2\cdot\cfrac{\frac{7}{2}}{1+\ddots}}}}\Rightarrow$$

$$2w_0=\cfrac{1}{1+\cfrac{3}{1+\cfrac{5}{1+\cfrac{7}{1+\ddots}}}}\quad\Rightarrow 1+2w_0=1+2\frac{I_1}{I_0}=1+\cfrac{1}{1+\cfrac{3}{1+\cfrac{5}{1+\cfrac{7}{1+\cfrac{9}{1+\ddots}}}}}$$

This proofs that the recursion $I_n=-2nI_n+2I_{n+1}+4I_{n+2}$ satisfies as our Continued Fraction and it evaluates to $1+2\dfrac{I_1}{I_0}$

At the other hand, the function $I_n = \int_0^\infty x^{n-\frac{1}{2}} e^{-x-x^2} dx$ also satisfies this recursive expression!

$$I_n = \int_0^\infty x^{n-\frac{1}{2}} e^{-x-x^2} dx = \int_0^\infty \frac{x^n}{\sqrt{x}} e^{-x-x^2} dx = 2\int_0^\infty x^n e^{-x-x^2} d\sqrt{x} = 2x^n e^{-x-x^2}\sqrt{x}\Big|_0^\infty - 2\int_0^\infty \sqrt{x} dx^n e^{-x-x^2}$$

$$= -2\int_0^\infty \sqrt{x} e^{-x-x^2}(-1-2x) + n\sqrt{x} x^{n-1} e^{-x-x^2} dx$$

$$= -2n\int_0^\infty x^{n-\frac{1}{2}} e^{-x-x^2} dx + 2\int_0^\infty x^{n+\frac{1}{2}} e^{-x-x^2} dx + 4\int_0^\infty x^{n+\frac{3}{2}} e^{-x-x^2} dx \Rightarrow$$

$$I_n = -2nI_n + 2I_{n+1} + 4I_{n+2}$$

Which proves theorem 14 that this Continued Fraction evaluates to the quotient of two integrals. The result cannot be expressed in closed form.

$$1+\cfrac{1}{1+\cfrac{3}{1+\cfrac{5}{1+\cfrac{7}{1+\cfrac{9}{1+\ddots}}}}} = 1+2\frac{I_1}{I_0} = 1+\frac{2\int_0^\infty x^{\frac{1}{2}} e^{-x-x^2} dx}{\int_0^\infty x^{-\frac{1}{2}} e^{-x-x^2} dx} \approx 1.46791991697366518..$$

Theorem 15

$$1+\cfrac{1}{1+\cfrac{2}{1+\cfrac{3}{1+\cfrac{4}{1+\ddots}}}}=\frac{1}{\sqrt{e}\int_1^{\infty}e^{-\frac{x^2}{2}}dx}$$

This is part of Ramanujan's more extensive theorem

$$\frac{1}{1}+\frac{1}{1\cdot 3}+\frac{1}{1\cdot 3\cdot 5}+\ldots\ldots \quad +\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{1+\cfrac{3}{1+\ddots}}}} \quad =\sqrt{\frac{\pi e}{2}}$$

Theorem 15 only deals with the Continued Fraction.

Proof:

Let $g(x)=e^{\frac{1}{2}x^2}\cdot\int_x^{\infty}e^{-\frac{1}{2}t^2}dt$. If we differentiate $g(x)$ we get:

$$g'(x)= \quad \frac{de^{\frac{1}{2}x^2}}{dx}\cdot\int_x^{\infty}e^{-\frac{1}{2}t^2}dt \quad + \quad e^{\frac{1}{2}x^2} \quad \frac{d\int_x^{\infty}e^{-\frac{1}{2}t^2}dt}{dx} \quad = \; x\mathrm{e}^{\frac{1}{2}x^2}\cdot\int_x^{\infty}e^{-\frac{1}{2}t^2}dt+e^{\frac{1}{2}x^2}\,\frac{dF(\infty)-dF(x)}{dx}$$

$$=x\cdot g(x)-1$$

Apparently the function $g(x)$ satisfies the differential equation $g'=x\cdot g-1$

Now we differentiate again and again and we establish the following pattern and recursive relation:

$$\begin{cases} g'' \;=x\cdot g-1 \\ g'' \;=g+xg' \\ g''' \;=2g'+xg'' \\ g'''' =3g''+xg''' \end{cases} \qquad \Rightarrow g^{(n+2)}=(n+1)g^{n}+xg^{(n+1)}$$

Dividing by $g^{(n+1)}$ results in $\frac{g^{(n+2)}}{g^{(n+1)}} = (n+1)\frac{g^{n}}{g^{(n+1)}} + x$

Now let: $r_n = \frac{g^{(n+1)}}{g^{(n)}} \Rightarrow r_{n+1} = \frac{g^{(n+2)}}{g^{(n+1)}}$ which results in the recursion $r_{n+1} = (n+1)\frac{1}{r_n} + x$

Therefore $r_n = \frac{(n+1)}{-x + r_{n+1}}$

With $g' = x \cdot g - 1$ and $r_0 = \frac{g'}{g}$ it follows that $r_0 = x - \frac{1}{g} \quad \Rightarrow g(x) = \frac{1}{x - r_0}$

This allows to express $g(x)$ as a Continued Fraction.

$$g(x) = \frac{1}{x - r_0} = \cfrac{1}{x - \cfrac{1}{-x + r_1}} = \cfrac{1}{x - \cfrac{1}{-x + \cfrac{2}{-x + r_2}}} = \cfrac{1}{x - \cfrac{1}{-x + \cfrac{2}{-x + \cfrac{3}{-x + r_3 \ddots}}}} \quad \Rightarrow$$

For $x = 1$ it follows that $g(1) = e^{\frac{1}{2}} \cdot \int_1^\infty e^{-\frac{1}{2}t^2}\, dt = \cfrac{1}{1 - \cfrac{1}{-1 + \cfrac{2}{-1 + \cfrac{3}{-1 + \ddots}}}}$

With some small manipulations we achieve theorem 15.

$$1 + \cfrac{1}{1 + \cfrac{2}{1 + \cfrac{3}{1 + \cfrac{4}{1 + \ddots}}}} = \frac{1}{\sqrt{e}\int_1^\infty e^{-\frac{x^2}{2}}\, dx} \approx\sim 1.525135276...$$

Theorem 16
$$0+\cfrac{1}{2+\cfrac{3}{4+\cfrac{5}{6+\cfrac{7}{8+\ddots}}}}=\frac{\int_0^1 x^{\frac{1}{2}}\cdot e^{\frac{1}{2}x}\,dx}{\int_0^1 x^{-\frac{1}{2}}\cdot e^{\frac{1}{2}x}\,dx}$$

Proof:

Let $w_n=\dfrac{I_{n+1}}{I_n}$ with $w_0=\cfrac{1}{2+\cfrac{3}{4+\cfrac{5}{6+\ddots}}}\Rightarrow w_n=\cfrac{1}{2+\cfrac{3}{4+\cfrac{5}{6+\cfrac{7}{8+\ddots}}}}=\dfrac{2n+1}{2n+2+w_{n+1}}$

This results in the recursive expression $(2n+1)I_n=(2n+2)I_{n+1}+I_{n+2}$

Applying '*Euler's Differential Method*' lemma 4, page 90, with
$a=1\quad \alpha=2\quad b=2\quad \beta=2\quad c=1\quad \gamma=0$ results in:

$$\frac{dS}{S}=\frac{-\frac{1}{2}dR}{R}+\frac{(2+2R)}{2(2R-2)}dR=\frac{-\frac{1}{2}dR}{R}+\frac{(R+1)}{2(R-1)}dR=\frac{-\frac{1}{2}dR}{R}+\frac{1}{2}dR+\frac{1}{R-1}dR\Rightarrow$$
$$\ln(S)=-\tfrac{1}{2}\ln(R)+\tfrac{1}{2}R+\ln(R-1)\quad\Rightarrow S=C\cdot R^{-\frac{1}{2}}\cdot e^{\frac{1}{2}R}\cdot|R-1|$$

Now chose $R(x)=x$ because this guarantees that $R^{n+1}(x)\cdot S(x)$ for $x=0$ and $x=1$.
$$I_0=\int_0^1 Pdx=\int\frac{S}{\beta R+\gamma R^2-\alpha}dR=\frac{1}{2}C\int_0^1\frac{x^{-\frac{1}{2}}\cdot e^{\frac{1}{2}x}\cdot(1-x)}{x-1}dx=-\frac{1}{2}C\int_0^1 x^{-\frac{1}{2}}\cdot e^{\frac{1}{2}x}\,dx \text{ en}$$
$$I_1=\int_0^1 PRdx=\int\frac{SR}{\beta R+\gamma R^2-\alpha}dR=\frac{1}{2}C\int_0^1\frac{x^{\frac{1}{2}}\cdot e^{\frac{1}{2}x}\cdot(1-x)}{x-1}dx=-\frac{1}{2}C\int_0^1 x^{\frac{1}{2}}\cdot e^{\frac{1}{2}x}\,dx$$

From this follows directly theorem 16.
$$w_0=\frac{I_1}{I_0}=0+\cfrac{1}{2+\cfrac{3}{4+\cfrac{5}{6+\cfrac{7}{8+\ddots}}}}=\frac{\int_0^1 x^{\frac{1}{2}}\cdot e^{\frac{1}{2}x}\,dx}{\int_0^1 x^{-\frac{1}{2}}\cdot e^{\frac{1}{2}x}\,dx}\approx 0.37973195474099563280...$$

Theorem 17

$$1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\ddots}}}}=\frac{\sum_{k=0}^{\infty}\frac{1}{k!k!}}{\sum_{k=0}^{\infty}\frac{1}{k!(k+1)!}}$$

Proof:

It is easy to show that this Continued fraction satisfies a recursive expression.

$$1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\ddots}}}} \Rightarrow w_0 = 1+\cfrac{1}{2+\cfrac{1}{3+\ddots}} \Rightarrow w_n = n+1+\frac{1}{w_{n+1}}$$

Let $w_n = \frac{I_n}{I_{n+1}}$ This allows to rewrite as $\frac{I_n}{I_{n+1}} = n+1+\frac{1}{\frac{I_{n+1}}{I_{n+2}}} \Rightarrow I_n = (n+1)I_{n+1}+I_{n+2}$

The series $I_n = \sum_{k=0}^{\infty}\frac{1}{k!\,(k+n)!}$ satisfies this recurrence because….

$$I_n-(n+1)I_{n+1} = \sum_{k=0}^{\infty}\frac{1}{k!\cdot(k+n)!}-\frac{n+1}{k!\cdot(k+n+1)!}=\sum_{k=0}^{\infty}\frac{k+n+1}{k!\cdot(k+n+1)!}-\frac{n+1}{k!\cdot(k+n+1)!}=$$

$$=\sum_{k=0}^{\infty}\frac{k}{k!\cdot(k+n+1)!}=\sum_{k=1}^{\infty}\frac{1}{(k-1)!\cdot(k+n+1)!}=\sum_{k^*=0}^{\infty}\frac{1}{k^*!\cdot(k^*+n+2)!}=I_{n+2}$$

Apparently the series $I_n = \sum_{k=0}^{\infty}\frac{1}{k!\,(k+n)!}$ satisfies $I_n = (n+1)I_{n+1}+I_{n+2}$

Now rewriting this recurrence results in $\frac{I_n}{I_{n+1}} = n+1+\frac{I_{n+2}}{I_{n+1}}$

Let $w_n = \frac{I_{n+1}}{I_n} = \frac{1}{n+1+w_{n+1}}$

This results in the Continued Fraction: $w_0 = \frac{1}{1+w_1} = \cfrac{1}{1+\cfrac{1}{2+w_2}} = \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+w_3}}}$

Which proves that: $\frac{1}{w_0} = \frac{I_0}{I_1} = 1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\ddots}}}} = \frac{\sum_{k=0}^{\infty}\frac{1}{k!\,k!}}{\sum_{k=0}^{\infty}\frac{1}{k!\,(k+1)!}} \approx 1.433127427$

The function $I_n = \sum_{k=0}^{\infty} \frac{1}{k!\,(k+n)!}$ is known as a *Bessel* function of the first kind.

Bessel functions are solutions of certain differential equations.

Besides as a series, *Bessel* functions can be defined as $I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(n\theta)d\theta$

Taking $z = 2$ results in a *modified Bessel*-function.

In particular $I_0(2)$ en $I_1(2)$ appear to satisfy the same recurrence as the continued fraction of theorem 17.

Lemma:

The integral $I_n(2) = \int_0^{\pi} e^{2\cos\theta}\cos(n\theta)d\theta$ satisfies $I_n = (n+1)I_{n+1} + I_{n+2}$

Proof:

$$I_n = \int_0^{\pi} e^{2\cos\theta}\cos(n\theta)d\theta$$

$$\begin{aligned}(n+1)I_{n+1} &= (n+1)\int_0^{\pi} e^{2\cos\theta}\cos((n+1)\theta)d\theta = \\ &= e^{2\cos\theta}\cdot\sin((n+1)\theta)\Big|_0^{\pi} + 2\int_0^{\pi} e^{2\cos\theta}\sin(\theta)\sin((n+1)\theta)d\theta = \\ &= 2\int_0^{\pi} e^{2\cos\theta}\sin(\theta)\sin(n\theta+\theta)d\theta\end{aligned}$$

$$I_n - (n+1)I_{n+1} = \int_0^{\pi} e^{2\cos\theta}\left[\cos(n\theta) - 2\sin(\theta)\sin(n\theta+\theta)\right]d\theta = \int_0^{\pi} e^{2\cos\theta}\cos(n\theta+2\theta)d\theta \Rightarrow$$

$$I_n = (n+1)I_{n+1} + I_{n+2}$$

Because

$$\begin{aligned}\cos(n\theta) - 2\sin(\theta)\sin(n\theta+\theta) &= \cos(n\theta) - 2\sin(\theta)\left[\sin(n\theta)\cos(\theta) + \cos(n\theta)\sin(\theta)\right] \\ &= \cos(n\theta) - \sin(2\theta)\sin(n\theta) - 2\sin^2(\theta)\cos(n\theta) \\ &= \cos(2\theta)\cos(n\theta) - \sin(2\theta)\sin(n\theta) \\ &= \cos(2\theta + n\theta) = \cos\left((n+2)\theta\right)\end{aligned}$$

Consequently, theorem 17 can also be expressed as

$$1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{6+\ddots}}}}}=\frac{I_0(2)}{I_1(2)}=\frac{\int_0^\pi e^{2\cos\theta}d\theta}{\int_0^\pi e^{2\cos\theta}\cos(\theta)d\theta}$$

It can also be shown more directly that *Bessel* functions can be expressed as both a series and as an integral expression.

The case *n=0*:

Lemma: $I_0=\sum_{k=0}^{\infty}\frac{1}{k!\,k!}=\frac{1}{\pi}\int_0^\pi e^{2\cos\theta}d\theta$

Proof:

$$\frac{1}{\pi}\int_0^\pi e^{2\cos(\theta)}d\theta=\frac{1}{\pi}\int_0^\pi\sum_{k=0}^{\infty}\frac{\left(2\cos\theta\right)^k}{k!}d\theta=\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{2^k}{k!}\int_0^\pi\cos^k(\theta)d\theta$$

Let $I_k=\int_0^\pi\cos^k(\theta)d\theta$

Apparently I_k satisfies a recurrence.

$$I_k=\int_0^\pi\cos^k(\theta)d\theta=(k-1)\int_0^\pi\cos^{k-2}(\theta)d\theta-(k-1)\int_0^\pi\cos^k(\theta)d\theta=$$

$$=(k-1)I_{k-2}-(k-1)k\cdot I_k\quad=\frac{k-1}{k}\cdot I_{k-2}$$

Since $I_0=\int_0^\pi d\theta=\pi$ and $I_1=\int_0^\pi\cos(\theta)d\theta=0$ it follows that

$$I_{1.3.5....}=0,\quad I_0=\pi,\quad I_2=\frac{1}{2}\pi,\quad I_4=\frac{3}{4}\cdot\frac{1}{2}\pi,\quad I_6=\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\pi$$

Now applying the summation:

$$\frac{1}{\pi}\int_0^\pi e^{2\cos(\theta)}d\theta=\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{2^k}{k!}I_k=\frac{1}{\pi}\left(\frac{2^0}{0!}I_0+\frac{2^2}{2!}I_2+\frac{2^4}{4!}I_4+....\right)=\frac{2^0}{0!}+\frac{2^2}{2!}\cdot\frac{1}{2}+\frac{2^4}{4!}\cdot\frac{3}{4}\cdot\frac{1}{2}+....$$

Writing out a number of terms reveals the pattern.

$$\frac{1}{\pi}\int_0^\pi e^{2\cos(\theta)}d\theta=\frac{1}{0!\cdot 0!}+\frac{1}{1!\cdot 1!}+\frac{1}{2!\cdot 2!}+\frac{1}{3!\cdot 3!}+\frac{1}{4!\cdot 4!}+.....$$

Which proves that $\frac{1}{\pi}\int_0^\pi e^{2\cos(\theta)}d\theta=\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{1}{k!}\qquad Q.E.D.$

The case *n=1*:

Lemma: $I_1=\sum_{k=0}^{\infty}\frac{1}{k!\,(k+1)!}=\frac{1}{\pi}\int_0^{\pi}e^{2\cos\theta}\cos(\theta)d\theta$

Proof:

By applying the series expansion of $e^{2\cos(\theta)}$ it follows:

$$\frac{1}{\pi}\int_0^{\pi}e^{2\cos(\theta)}\cos(\theta)d\theta=\frac{1}{\pi}\int_0^{\pi}\sum_{k=0}^{\infty}\frac{(2\cos\theta)^k}{k!}\cos(\theta)d\theta$$

Interchanging the summation and the integral results in

$$\frac{1}{\pi}\int_0^{\pi}e^{2\cos(\theta)}\cos(\theta)d\theta=\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{2^k}{k!}\int_0^{\pi}\cos^{k+1}(\theta)d\theta$$

Let $I_k=\int_0^{\pi}\cos^{k+1}(\theta)d\theta$ This reduction integral can be rewritten as a recurrence

$$I_k=\int_0^{\pi}\cos^{k+1}(\theta)d\theta\quad\Rightarrow I_k=k\int_0^{\pi}\cos^{k-1}(\theta)d\theta-k\int_0^{\pi}\cos^{k+1}(\theta)d\theta=k\cdot I_{k-2}-k\cdot I_k$$

Therefore $I_k=\frac{k}{k+1}\cdot I_{k-2}$ and since $I_0=\int_0^{\pi}\cos(\theta)d\theta=0$ it follows that $I_{0,2,4,6...}=0$

Since $I_1=\int_0^{\pi}\cos^2(\theta)d\theta=\frac{1}{2}\pi$, $I_3=\frac{3}{4}\cdot\frac{1}{2}\pi\quad I_5=\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\pi\quad$ etc

Now the summation is applied and by writing our a number of terms the pattern becomes obvious.

$$\begin{aligned}\frac{1}{\pi}\int_0^{\pi}e^{2\cos(\theta)}\cos(\theta)d\theta&=\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{2^k}{k!}I_k=\frac{1}{\pi}\cdot\left[\frac{2^1}{1!}I_1+\frac{2^3}{3!}I_3+\frac{2^5}{5!}I_5+\frac{2^7}{7!}I_7+...\right]\\&=\frac{2^1}{1!}\cdot\frac{1}{2}+\frac{2^3}{3!}\cdot\frac{3}{4}\cdot\frac{1}{2}+\frac{2^5}{5!}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}+\frac{2^7}{7!}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}+...\\&=\frac{1}{0!\cdot1!}+\frac{1}{1!\cdot2!}+\frac{1}{2!\cdot3!}+\frac{1}{3!\cdot4!}+\frac{1}{4!\cdot5!}+....\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{1}{(k+1)!}\qquad Q.E.D.\end{aligned}$$

The case *n=2*:

Lemma: $I_2=\sum_{k=0}^{\infty}\frac{1}{k!\,(k+2)!}=\frac{1}{\pi}\int_0^{\pi}e^{2\cos\theta}\cos(2\theta)d\theta$

Proof:

For *n=2* the identity $\cos(2\theta)=2\cos^2(\theta)-1$ is useful.

Again the series expansion of $e^{2\cos(\theta)}$ and the exchange of summation and integral is used. $\frac{1}{\pi}\int_0^{\pi}e^{2\cos(\theta)}\cos(2\theta)d\theta=\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{2^k}{k!}\int_0^{\pi}2\cos^{k+2}(\theta)-\cos^k(\theta)d\theta$

The integrals are familiar. Let $I_k=\int_0^{\pi}\cos^k(\theta)d\theta$ with the recurrence $I_{k+2}=\frac{k+1}{k+2}\cdot I_k$

Now consider $W_k=\int_0^{\pi}2\cos^{k+2}(\theta)-\cos^k(\theta)d\theta$.

W_k satisfies a recurrence relation: $W_k=2I_{k+2}-I_k=2\frac{k+1}{k+2}\cdot I_k-I_k\Rightarrow W_k=\frac{k}{k+2}\cdot I_k$

I_0 is easy to evaluate: $I_0=\int_0^{\pi}d\theta=\pi$

Using $I_{k+2}=\frac{k+1}{k+2}\cdot I_k$ it follows that $I_2=\frac{1}{2}\pi,\quad I_4=\frac{3}{4}\cdot\frac{1}{2}\pi,\quad I_6=\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\pi$ etc.

Because $W_k=\frac{k}{k+2}\cdot I_k$, it follows that $W_0=0\cdot I_0,\ W_2=\frac{2}{4}\cdot I_2,\ W_4=\frac{4}{6}\cdot I_4,\ W_6=\frac{6}{8}\cdot I_6$

Now the summation is applied.

$$\begin{aligned}\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{2^k}{k!}W_k\quad&=\frac{2^0}{0!}\cdot 0+\frac{2^2}{2!}\cdot\frac{2}{4}\cdot\frac{1}{2}+\frac{2^4}{4!}\cdot\frac{4}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}+\frac{2^6}{6!}\cdot\frac{6}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}+\frac{2^8}{8!}\cdot\frac{8}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}+...\\&=\frac{1}{0!\cdot 2!}+\frac{1}{1!\cdot 3!}+\frac{1}{2!\cdot 4!}+\frac{1}{3!\cdot 5!}+\frac{1}{4!\cdot 6!}+....\end{aligned}$$

Instead of evaluating and rewriting terms, also a more algebraic approach is possible.

Applying $I_k=\frac{k!}{2^k\cdot\frac{k}{2}!^2}\cdot\pi$ for $k=0,2,4,...$ in the summation results in:

$$\begin{aligned}\frac{1}{\pi}\int_0^{\pi}e^{2\cos(\theta)}\cos(2\theta)d\theta&=\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{2^k}{k!}W_k=\frac{1}{\pi}\sum_{k=2,4,6...}^{\infty}\frac{2^k}{k!}\cdot\frac{k}{k+2}\cdot\frac{k!}{2^k\cdot\frac{k}{2}!^2}\cdot\pi\ =\sum_{k=2,4,6...}^{\infty}\frac{\frac{k}{2}}{\frac{k}{2}+1}\frac{1}{\cdot\frac{k}{2}!^2}\\&=\sum_{k=2,4,6...}^{\infty}\frac{1}{\left(\frac{k}{2}-1\right)!\cdot\left(\frac{k}{2}+1\right)!}=\sum_{k=0,1,2}^{\infty}\frac{1}{k!}\cdot\frac{1}{(k+2)!}\end{aligned}$$

Which completes the proof that $\frac{1}{\pi}\int_0^{\pi}e^{2\cos(\theta)}\cos(2\theta)d\theta=\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{1}{(k+2)!}\quad Q.E.D.$

The case *n=3*:

Lemma: $I_3 = \sum_{k=0}^{\infty} \frac{1}{k!\,(k+3)!} = \frac{1}{\pi}\int_0^{\pi} e^{2\cos\theta}\cos(3\theta)d\theta$

Proof:

For *n=3* the identity $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$ is useful.

Applying the series expansion results in

$$\frac{1}{\pi}\int_0^{\pi} e^{2\cos(\theta)}\cos(3\theta)d\theta = \frac{1}{\pi}\int_0^{\pi}\sum_{k=0}^{\infty}\frac{(2\cos\theta)^k}{k!}\cos(3\theta)d\theta = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{2^k}{k!}\int_0^{\pi} 4\cos^{k+3}(\theta) - 3\cos^{k+1}(\theta)d\theta$$

Let $W_k = \int_0^{\pi} 4\cos^{k+3}(\theta) - 3\cos^{k+1}(\theta)d\theta$ and let $I_k = \int_0^{\pi}\cos^k(\theta)d\theta$

W_k can be expressed as the recurrence $W_k = 4I_{k+3} - 3I_{k+1}$

With $I_0 = \pi$ and $I_2 = \frac{1}{2}\cdot\pi \quad I_4 = \frac{3}{4}\cdot\frac{1}{2}\cdot\pi \quad I_6 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\pi$ it follows that

$$W_0 = 0 \quad W_1 = 0 \quad W_2 = 0 \quad W_3 = \frac{1}{8}\cdot\pi \quad W_5 = \frac{5}{32}\cdot\pi \quad W_7 = \frac{21}{128}\cdot\pi \ .....$$

Now applying the summation results in:

$$\frac{1}{\pi}\int_0^{\pi}\sum_{k=0}^{\infty}\frac{(2\cos\theta)^k}{k!}\cos(3\theta)d\theta = \frac{2^3}{3!}\cdot\frac{1}{8} + \frac{2^5}{5!}\cdot\frac{5}{32} + \frac{2^7}{7!}\cdot\frac{21}{128} + ... = \frac{1}{0!\cdot 3!} + \frac{1}{1!\cdot 4!} + \frac{1}{2!\cdot 5!} + ...$$

Which completes the proof that $\frac{1}{\pi}\int_0^{\pi} e^{2\cos(\theta)}\cos(3\theta)d\theta = \sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{1}{(k+3)!}$ *Q.E.D.*

The case *n*=*n*:
A *direct* general proof for *n* appears to be difficult.

Lemma: $I_n = \sum_{k=0}^{\infty} \frac{1}{k!\,(k+n)!} = \frac{1}{\pi}\int_0^{\pi} e^{2\cos\theta} \cos(n\theta) d\theta$

Proof:

The series $\sum_{k=0}^{\infty} \frac{1}{k!\,(k+n)!}$ and the integral $I_n = \int_0^{\pi} e^{2\cos\theta} \cos(n\theta) d\theta$ satisfy the same recurrence relation: $I_n = \left(n+1\right) I_{n+1} + I_{n+2}$ (Proven on page 68 and 70)

Therefore, both expressions must be similar, apart from a constant factor.

$$C \cdot \int_0^{\pi} e^{2\cos\theta} \cos(n\theta) d\theta = \sum_{k=0}^{\infty} \frac{1}{k!\,(k+n)!}$$

Determining the constant *C* can be done by evaluating the situation for $n=0$

For $n=0$ is was already found that $\sum_{k=0}^{\infty} \frac{1}{k!\,k!} = \frac{1}{\pi}\int_0^{\pi} e^{2\cos\theta} d\theta$

Therefore, $C = \frac{\sum_{k=0}^{\infty} \frac{1}{k!\,k!}}{\int_0^{\pi} e^{2\cos\theta} d\theta} = \frac{\frac{1}{\pi}\int_0^{\pi} e^{2\cos\theta} d\theta}{\int_0^{\pi} e^{2\cos\theta} d\theta} = \frac{1}{\pi}$

Which proves that $\frac{1}{\pi}\int_0^{\pi} e^{2\cos\theta} \cos(n\theta) d\theta = \sum_{k=0}^{\infty} \frac{1}{k!\,(k+n)!}$

Some Continued Fractions resulting in special functions

Theorem 18

$$1+\cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{3}}{1+\cfrac{\frac{1}{5}}{1+\cfrac{\frac{1}{7}}{1+\cfrac{\frac{1}{9}}{1+\frac{1}{11}\ddots}}}}}=\cfrac{2}{1+\cfrac{I_1\left(\frac{1}{4}\right)}{I_0\left(\frac{1}{4}\right)}}$$

Where $I_0\left(\frac{1}{4}\right)$ and $I_1\left(\frac{1}{4}\right)$ are "*Modified Bessel-functions of the first kind*",

defined as: $I_0\left(\frac{1}{4}\right)=\frac{1}{\pi}\int_0^{\pi} e^{\frac{1}{4}\cos\theta}d\theta$ and $I_1\left(\frac{1}{4}\right)=\frac{1}{\pi}\int_0^{\pi} e^{\frac{1}{4}\cos\theta}\cos(\theta)d\theta$

Proof:

$\cfrac{1}{1+\cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{3}}{1+\cfrac{\frac{1}{5}}{1+\ddots}}}}$ can be rewritten as a recurrence $w_n=\cfrac{1}{1+\cfrac{w_{n+1}}{2n+1}}$

Which can be rewritten as: $2n+1+w_{n+1}=\frac{2n+1}{w_n}$

Now assume $w_n=\frac{I_{n+1}}{I_n}$ which results in the recurrence $(2n+1)I_n=(2n+1)I_{n+1}+I_{n+2}$

Applying '*Euler's Differential Method*' lemma 4, page 90 with
$a=1 \quad \alpha=2 \quad b=1 \quad \beta=2 \quad c=1 \quad \gamma=0$ results in the differential equation:

$$\frac{dS}{S}=-\frac{\frac{1}{2}}{R}dR+\frac{R}{2(R-1)}dR \Rightarrow$$

$$\ln(S)=-\tfrac{1}{2}\ln(|R|)+\tfrac{1}{2}R+\tfrac{1}{2}\ln(|R-1|)\Rightarrow S=C\cdot e^{\frac{1}{2}R}R^{-\frac{1}{2}}\cdot\left(|R-1|\right)^{\frac{1}{2}}$$

The condition $S\cdot R^{n+1}=0$ for $x=0$ and $\text{en}\, x=1$ is satisfied when $R(x)=x$

Therefore the solution of the differential equation is: $S=C\cdot e^{\frac{1}{2}x}x^{-\frac{1}{2}}\cdot(1-x)^{\frac{1}{2}}$

Now I_0 and I_1 can be calculated.

$$I_0 = \int_0^1 P dx = \int_0^1 \frac{S}{2R-2} dR = -\tfrac{1}{2} C \int_0^1 e^{\frac{1}{2}x} \cdot \frac{x^{-\frac{1}{2}}}{(1-x)^{\frac{1}{2}}} dx$$

$$I_1 = \int_0^1 PR dx = \int_0^1 \frac{SR}{2R-2} dR = -\tfrac{1}{2} C \int_0^1 e^{\frac{1}{2}x} \cdot \frac{x^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}} dx$$

These integrals can be rewritten by the substitution $x = \dfrac{1+\cos(\theta)}{2}$

We can express I_0 as:

$$I_0 = \tfrac{1}{2} \int_0^{\pi} e^{\frac{1}{4}(1+\cos(\theta))} \cdot \frac{\sqrt{2} \cdot \sqrt{2}}{(1+\cos(\theta))^{\frac{1}{2}} \cdot (1-\cos(\theta))^{\frac{1}{2}}} \cdot \sin(\theta) d\theta = e^{\frac{1}{4}} \int_0^{\pi} e^{\frac{1}{4}\cos(\theta)} d\theta$$

We can express I_1 as:

$$I_1 = \tfrac{1}{2} \int_0^{\pi} e^{\frac{1}{4}(1+\cos(\theta))} \cdot \frac{\sqrt{2} \cdot (1+\cos(\theta))^{\frac{1}{2}}}{\sqrt{2} \cdot (1-\cos(\theta))^{\frac{1}{2}}} \cdot \sin(\theta) d\theta = \tfrac{1}{2} e^{\frac{1}{4}} \int_0^{\pi} e^{\frac{1}{4}\cos(\theta)} \cdot (1+\cos(\theta)) d\theta$$

Par definition, $I_0\left(\tfrac{1}{4}\right) = \int_0^{\pi} e^{\frac{1}{4}\cos(\theta)} \cdot d\theta$ and $I_1\left(\tfrac{1}{4}\right) = \int_0^{\pi} e^{\frac{1}{4}\cos(\theta)} \cdot \cos(\theta) d\theta$

Therefore: $\dfrac{1}{1+\dfrac{\frac{1}{1}}{1+\dfrac{\frac{1}{3}}{1+\dfrac{\frac{1}{5}}{1+\ddots}}}} = \dfrac{I_1}{I_0} = \dfrac{\frac{1}{2} e^{\frac{1}{4}} \left(I_0\left(\frac{1}{4}\right) + I_1\left(\frac{1}{4}\right)\right)}{e^{\frac{1}{4}} I_0\left(\frac{1}{4}\right)} = \dfrac{I_0\left(\frac{1}{4}\right) + I_1\left(\frac{1}{4}\right)}{2 I_0\left(\frac{1}{4}\right)}$

By reversing we now find the value of the Continued Fraction:

$$1 + \cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{3}}{1+\cfrac{\frac{1}{5}}{1+\cfrac{\frac{1}{7}}{1+\cfrac{\frac{1}{9}}{1+\frac{1}{11}\ddots}}}}} = \frac{2}{1+\dfrac{I_1\left(\frac{1}{4}\right)}{I_0\left(\frac{1}{4}\right)}} \approx 1.77930639663980147122766 47\ldots \quad \text{Q.E.D.}$$

Without proof, this Continued Fraction can also be evaluated by applying the so-called "*Confluent HyperGeometric function*".

Lemma: $$1+\cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{3}}{1+\cfrac{\frac{1}{5}}{1+\cfrac{\frac{1}{7}}{1+\cfrac{\frac{1}{9}}{1+\frac{1}{11}\ddots}}}}}=\frac{2F_1\left(\frac{1}{2},1,\frac{1}{2}\right)}{F_1\left(\frac{3}{2},1,\frac{1}{2}\right)}$$

These HyperGeometric functions are defined as a series.

$$2F_1\left(\tfrac{1}{2},1,\tfrac{1}{2}\right)=\sum_{k=0}^{\infty}\frac{(2k)!}{k!^3\,2^{3k-1}}\qquad=\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{k!\cdot 2^{3k-1}}$$

and

$$F_1\left(\tfrac{3}{2},1,\tfrac{1}{2}\right)=\sum_{k=0}^{\infty}\frac{(2k+2)!}{k!(k+1)!^2\;2^{3k+1}}=\sum_{k=0}^{\infty}\frac{\binom{2k+2}{k+1}}{k!\;2^{3k+1}}$$

This allows to express the Continued Fraction as the quotient of series:

$$1+\cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{3}}{1+\cfrac{\frac{1}{5}}{1+\cfrac{\frac{1}{7}}{1+\cfrac{\frac{1}{9}}{1+\frac{1}{11}\ddots}}}}}=\frac{2}{1+\frac{I_1(\frac{1}{4})}{I_0(\frac{1}{4})}}=\frac{2F_1(\frac{1}{2},1,\frac{1}{2})}{F_1(\frac{3}{2},1,\frac{1}{2})}=\frac{\displaystyle\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{k!\cdot 2^{3k-1}}}{\displaystyle\sum_{k=0}^{\infty}\frac{\binom{2k+2}{k+1}}{k!\;2^{3k+1}}}\approx 1.7793...$$

Theorem 19

$$\frac{1}{1}+\cfrac{1}{\frac{1}{3}+\cfrac{1}{\frac{1}{5}+\cfrac{1}{\frac{1}{7}+\cfrac{1}{\frac{1}{9}+\cfrac{1}{\frac{1}{11}+\ddots}}}}}=\frac{\frac{1}{8}\left(\frac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})}\right)^2+1}{\frac{1}{8}\left(\frac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})}\right)^2-1}$$

Where $\Gamma(p)$ is the "Gamma *functions"*, defined by $\Gamma(p)=\int\limits_0^\infty x^{p-1}\,e^{-x}dx$

Proof:

Let $w_n=\cfrac{1}{\frac{1}{1}+\cfrac{1}{\frac{1}{3}+\cfrac{1}{\frac{1}{5}+\cfrac{1}{\frac{1}{7}+\cfrac{1}{\frac{1}{9}+\cfrac{1}{\frac{1}{11}+\ddots}}}}}=\cfrac{1}{\frac{1}{2n+1}+w_{n+1}}$

This can be rewritten as the recurrence $1+(2n+1)\,w_{n+1}=\dfrac{2n+1}{w_n}$

Now let $w_n=\dfrac{I_{n+1}}{I_n}$. This results in the recurrence $(2n+1)\,I_n=I_{n+1}+(2n+1)\,I_{n+2}$

Applying '*Euler's Differential Method*' lemma 4, page 90 with

$a=1 \quad \alpha=2 \quad b=1 \quad \beta=0 \quad c=1 \quad \gamma=2$ results in the differential equation:

$\dfrac{dS}{S}=-\dfrac{\frac{1}{2}}{R}dR+\dfrac{R}{2(R^2-1)}dR$ which has as its general solution

$$\ln(S)=C+-\tfrac{1}{2}\ln(|R|)+\tfrac{1}{4}\ln\left(\frac{|1-R|}{|1+R|}\right)\;\Rightarrow\; S=C\cdot R^{-\frac{1}{2}}\cdot\left(\frac{|1-R|}{|1+R|}\right)^{\frac{1}{4}}$$

Again the condition $S\cdot R^{n+1}=0$ for $x=0$ and $\mathrm{en}\,x=1$ is satisfied when $R(x)=x$

Therefore the solution of the differential equation is: $S=C\cdot x^{-\frac{1}{2}}\cdot\left(\dfrac{|1-x|}{|1+x|}\right)^{\frac{1}{4}}$

We can now evaluate I_0 and I_1

$$\begin{cases} I_0 = \int_0^1 P dx = \tfrac{1}{2}\int_0^1 \frac{S}{R^2-1}dx = \tfrac{1}{2}C\int_0^1 \frac{x^{-\frac{1}{2}}\left(\frac{1-x}{1+x}\right)^{\frac{1}{4}}}{x^2-1}dx \\ I_1 = \int_0^1 PR dx = \tfrac{1}{2}\int_0^1 \frac{SR}{R^2-1}dx = \tfrac{1}{2}C\int_0^1 \frac{x^{\frac{1}{2}}\left(\frac{1-x}{1+x}\right)^{\frac{1}{4}}}{x^2-1}dx \end{cases}$$

Applying the substitution

$$t = \frac{1-x}{1+x} \Rightarrow x = \frac{1-t}{1+t} \Rightarrow \quad dx = \frac{-2}{(1+t)^2}dt \quad \text{and} \quad 1-x^2 = \frac{4t}{(1+t)^2}$$

results after careful manipulation in:

$$\begin{cases} I_0 = \tfrac{1}{4}C\int_0^1 \left(\frac{1+t}{1-t}\right)^{\frac{1}{2}} t^{-\frac{3}{4}}dt = C^*\int_0^1 \left(\frac{1+t}{1-t}\right)^{\frac{1}{2}} t^{-\frac{3}{4}}dt \\ I_1 = \tfrac{1}{4}C\int_0^1 \left(\frac{1-t}{1+t}\right)^{\frac{1}{2}} t^{-\frac{3}{4}}dt = C^*\int_0^1 \left(\frac{1-t}{1+t}\right)^{\frac{1}{2}} t^{-\frac{3}{4}}dt \end{cases}$$

Therefore $\dfrac{1}{w_0} = \dfrac{I_0}{I_1} = \tfrac{1}{1} + \cfrac{1}{\tfrac{1}{3} + \cfrac{1}{\tfrac{1}{5} + \cfrac{1}{\tfrac{1}{7} + \cfrac{1}{\tfrac{1}{9} + \cfrac{1}{\tfrac{1}{11} + \ddots}}}}} = \dfrac{\int_0^1 \frac{(1+t)^{\frac{1}{2}}}{(1-t)^{\frac{1}{2}}}\cdot t^{-\frac{3}{4}}dt}{\int_0^1 \frac{(1-t)^{\frac{1}{2}}}{(1+t)^{\frac{1}{2}}}\cdot t^{-\frac{3}{4}}dt}$

This quotient of integral expressions can be converted towards a quotient of Gamma functions. The clever trick used was found by Euler.

$$\begin{cases} I_0 = C^*\int_0^1 \frac{(1+t)^{\frac{1}{2}}}{(1-t)^{\frac{1}{2}}} t^{-\frac{3}{4}}dt \ = C^*\int_0^1 \frac{(1+t)}{(1-t^2)^{\frac{1}{2}}} t^{-\frac{3}{4}}\,dt = C^*\int_0^1 (1-t^2)^{-\frac{1}{2}}\left(t^{-\frac{3}{4}} + t^{\frac{1}{4}}\right)dt \\ I_1 = C^*\int_0^1 \frac{(1-t)^{\frac{1}{2}}}{(1+t)^{\frac{1}{2}}} t^{-\frac{3}{4}}dt \ = C^*\int_0^1 \frac{(1-t)}{(1-t^2)^{\frac{1}{2}}} t^{-\frac{3}{4}}\,dt = C^*\int_0^1 (1-t^2)^{-\frac{1}{2}}\left(t^{-\frac{3}{4}} - t^{\frac{1}{4}}\right)dt \end{cases}$$

Now assume $t^2 = x \Rightarrow t = \sqrt{x} \Rightarrow dt = \dfrac{1}{2\sqrt{x}}dx$

After some manipulation we find:

$$\begin{cases} I_0 = \tfrac{1}{2}C^* \int_0^1 (1-x)^{-\frac{1}{2}} \left(x^{-\frac{3}{8}} + x^{\frac{1}{8}}\right) \dfrac{1}{x^{\frac{1}{2}}} dx = \tfrac{1}{2}C^* \int_0^1 (1-x)^{-\frac{1}{2}} (x^{-\frac{7}{8}} + x^{-\frac{3}{8}}) dx \\ I_1 = \tfrac{1}{2}C^* \int_0^1 (1-x)^{-\frac{1}{2}} \left(x^{-\frac{3}{8}} - x^{\frac{1}{8}}\right) \dfrac{1}{x^{\frac{1}{2}}} dx = \tfrac{1}{2}C^* \int_0^1 (1-x)^{-\frac{1}{2}} \left(x^{-\frac{7}{8}} - x^{\frac{3}{8}}\right) dx \end{cases}$$

Obviously, for the evaluation of the quotient of these integrals, the factor $\tfrac{1}{2}C^*$ can be ignored.

Now the identity $B(p,q) = \int_0^1 (1-x)^{p-1} \cdot x^{q-1}\, dx = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ is applied.

$$\begin{cases} I_0 = \int_0^1 (1-x)^{-\frac{1}{2}} \cdot x^{-\frac{7}{8}} dx \quad + \quad \int_0^1 (1-x)^{-\frac{1}{2}} \cdot x^{-\frac{3}{8}} dx \\ I_1 = \int_0^1 (1-x)^{-\frac{1}{2}} \cdot x^{-\frac{7}{8}} dx \quad - \quad \int_0^1 (1-x)^{-\frac{1}{2}} \cdot x^{-\frac{3}{8}} dx \end{cases} \Rightarrow p = \tfrac{1}{2} \;\; q = \tfrac{5}{8} \;\Rightarrow$$

$$\begin{cases} I_0 = \dfrac{\Gamma(\frac{1}{2}) \cdot \Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})} + \dfrac{\Gamma(\frac{1}{2}) \cdot \Gamma(\frac{5}{8})}{\Gamma(\frac{9}{8})} = \Gamma(\tfrac{1}{2}) \cdot \left[\dfrac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})} + \dfrac{\Gamma(\frac{5}{8})}{\Gamma(\frac{9}{8})} \right] \\ I_1 = \dfrac{\Gamma(\frac{1}{2}) \cdot \Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})} - \dfrac{\Gamma(\frac{1}{2}) \cdot \Gamma(\frac{5}{8})}{\Gamma(\frac{9}{8})} = \Gamma(\tfrac{1}{2}) \cdot \left[\dfrac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})} + \dfrac{\Gamma(\frac{5}{8})}{\Gamma(\frac{9}{8})} \right] \end{cases}$$

Therefore $\dfrac{I_0}{I_1} = \dfrac{\left[\dfrac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})} + \dfrac{\Gamma(\frac{5}{8})}{\Gamma(\frac{9}{8})} \right]}{\left[\dfrac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})} - \dfrac{\Gamma(\frac{5}{8})}{\Gamma(\frac{9}{8})} \right]} = \dfrac{\left[\dfrac{\Gamma(\frac{1}{8})\Gamma(\frac{9}{8})}{\left(\Gamma(\frac{5}{8})\right)^2} + 1 \right]}{\left[\dfrac{\Gamma(\frac{1}{8})\Gamma(\frac{9}{8})}{\left(\Gamma(\frac{5}{8})\right)^2} - 1 \right]} = \dfrac{\frac{1}{8}\left(\dfrac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})} \right)^2 + 1}{\frac{1}{8}\left(\dfrac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})} \right)^2 - 1}$

In the last step, the property of the Gamma function $p \cdot \Gamma(p) = \Gamma(p+1)$ is used.

Which finishes the proof that

$$\frac{1}{1} + \cfrac{1}{\frac{1}{3} + \cfrac{1}{\frac{1}{5} + \cfrac{1}{\frac{1}{7} + \cfrac{1}{\frac{1}{9} + \cfrac{1}{\frac{1}{11} + \ddots}}}}} = \frac{\frac{1}{8}\left(\dfrac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})} \right)^2 + 1}{\frac{1}{8}\left(\dfrac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})} \right)^2 - 1} \approx 1.81706040611130748358$$

Some 'Mirrored' Continued Fractions

Theorem 5

$$\frac{1}{1}+\cfrac{1}{\frac{1}{2}+\cfrac{1}{\frac{1}{3}+\cfrac{1}{\frac{1}{4}+\cfrac{1}{\frac{1}{5}+\ddots}}}}=\frac{2}{\pi-2}$$

Theorem 9.2

$$1+\cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{2}}{1+\cfrac{\frac{1}{3}}{1+\cfrac{\frac{1}{4}}{1+\ddots}}}}=e-1$$

Theorem 5 and theorem 9.2, placed side by side, produce a beautiful "mirrored" image. They seem to have a great resemblance. But that's just visual. Both the outcome and the evidence have no relation to each other. For the proof of theorem 5 and theorem 9.2 see pages 32 & 48

Theorem 12

$$1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\ddots}}}}=\frac{e^2+1}{e^2-1}$$

Theorem 14

$$1+\cfrac{1}{1+\cfrac{3}{1+\cfrac{5}{1+\cfrac{7}{1+\ddots}}}}=1+\frac{2\int_0^\infty x^{\frac{1}{2}}e^{-x-x^2}dx}{\int_0^\infty x^{-\frac{1}{2}}e^{-x-x^2}dx}$$

Also the theorems 12 and 14 provide in a beautifully mirrored image that suggests a connection. But it's only visual. For the proof of theorem 12 and theorem 14 see pages 58 & 63.

Theorem 9.3

$$1+\cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{7+\cfrac{8}{9+\ddots}}}}=\frac{1}{\sqrt{e}-1}$$

Theorem 16

$$0+\cfrac{1}{2+\cfrac{3}{4+\cfrac{5}{6+\cfrac{7}{8+\ddots}}}}=\frac{\int_0^1 x^{\frac{1}{2}}\cdot e^{\frac{1}{2}x}\,dx}{\int_0^1 x^{-\frac{1}{2}}\cdot e^{\frac{1}{2}x}\,dx}$$

Also theorem 9.3 and theorem 16 show this "mirroring". But again, it is only a visual relation. For proofs of theorem 9.3 and 16, see pages 48 and 67.

Theorem 15

$$1+\cfrac{1}{1+\cfrac{2}{1+\cfrac{3}{1+\cfrac{4}{1+\ddots}}}}=\frac{1}{\sqrt{e}\int_1^\infty e^{-\frac{x^2}{2}}dx}$$

Theorem 17

$$1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\ddots}}}}=\frac{I_0(2)}{I_1(2)}$$

Also theorem 15 and 17 display a visually beautiful pattern. But there is no connection and the evaluation leads towards two entirely different results.
For proofs, see pages 65 & 68

Theorem 18

$$1+\cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{3}}{1+\cfrac{\frac{1}{5}}{1+\cfrac{\frac{1}{7}}{1+\ddots}}}}=\frac{2}{1+\frac{I_1(\frac{1}{4})}{I_0(\frac{1}{4})}}$$

Theorem 19

$$\frac{1}{1}+\cfrac{1}{\frac{1}{3}+\cfrac{1}{\frac{1}{5}+\cfrac{1}{\frac{1}{7}+\cfrac{1}{\frac{1}{9}+\ddots}}}}=\frac{\frac{1}{8}\left(\frac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})}\right)^2+1}{\frac{1}{8}\left(\frac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})}\right)^2-1}$$

Another example. Also the “Harmonic ” Continued Fraction of theorem 18 has a mirrored version, again with a totally different result.
For proofs of theorem 18 en theorem 19 see page 75 & 78.

Theorem 20

$$1+\cfrac{1^2}{1+\cfrac{2^2}{1+\cfrac{3^2}{1+\cfrac{4^2}{1+\ddots}}}}=\frac{1}{\ln(2)}$$

Proof:

$\ln(2)$ can be expressed as the integral $\int_0^1 \frac{1}{1+x}dx$.

The integral $\int_0^1 \frac{x^{n-1}}{1+x^m}dx$ can be written as a Continued Fraction by lemma 3, page 89.

$$\int_0^1 \frac{x^{n-1}}{1+x^m}dx=\cfrac{1}{n+\cfrac{n^2}{m+\cfrac{(m+n)^2}{m+\cfrac{(2m+n)^2}{m+\cfrac{(3m+n)^2}{m+\ddots}}}}}$$

Choosing $m=1$ and $n=1$ results directly in theorem 20.

$$\ln(2)=\cfrac{1}{1+\cfrac{1^2}{1+\cfrac{2^2}{1+\cfrac{3^2}{1+\cfrac{4^2}{1+\ddots}}}}} \qquad \text{Q.E.D.}$$

An alternative proof uses lemma 5, page 92: $\dfrac{1}{s+\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{n^2}{s}\right)} = 2\displaystyle\int_0^1 \frac{x^s}{1+x^2}\,dx$ which allows to generate a couple of new Continued Fractions.

Choosing $s=1$ results in: $$1+\cfrac{1^2}{1+\cfrac{2^2}{1+\cfrac{3^2}{1+\cfrac{4^2}{1+\ddots}}}}=\frac{1}{\ln(2)}$$

Choosing $s=3$ results in: $$3+\cfrac{1^2}{3+\cfrac{2^2}{3+\cfrac{3^2}{3+\cfrac{4^2}{3+\ddots}}}}=\frac{1}{1-\ln(2)}$$

Choosing $s=5$ results in: $$5+\cfrac{1^2}{5+\cfrac{2^2}{5+\cfrac{3^2}{5+\cfrac{4^2}{5+\ddots}}}}=\frac{2}{2\ln(2)-1}$$ etc.

These Continued Fractions are nothing more than a slight variation of theorem 20. Which is a nice Continued Fraction with a compact result and a simple elementary proof. What a contrast to the next very last Theorem…

Theorem 21

$$1^2 + \cfrac{1}{2^2 + \cfrac{1}{3^2 + \cfrac{1}{4^2 + \cfrac{1}{5^2 + \cfrac{1}{6^2 + \ddots}}}}} = \ ?$$

Theorem 21 seems like nothing but a slight variation on the same theme.
But the contrary is true….

Thousands of decimals are known. To list a few,

$$1^2 + \cfrac{1}{2^2 + \cfrac{1}{3^2 + \cfrac{1}{4^2 + \cfrac{1}{5^2 + \cfrac{1}{6^2 + \ddots}}}}} \approx 1.24328847839971564408249654 5......$$

The recursive equation is easy to find: $w_n = \dfrac{1}{n^2 + w_{n+1}}$

Let $w_n = \dfrac{I_{n+1}}{I_n}$ which results in the recursive expression $I_n = n^2 I_{n+1} + I_{n+2}$

But then ….. Every previously described method fails!
Euler's Differential Method is not applicable.
Also studying the resulting fractions does not help although it reveals a recursive pattern in the numerator and denominator:

$w_1 = \dfrac{n^2 a_{n-1} + a_{n-2}}{n^2 b_{n-1} + b_{n-2}}$ with $a_1 = 1,\ a_2 = 5,\ b_1 = 1,\ b_2 = 4$ resulting in

$w_1 = \dfrac{1}{1}, w_2 = \dfrac{5}{4}, w_3 = \dfrac{46}{37}, w_4 = \dfrac{741}{596}, w_5 = \dfrac{18571}{14937}, w_6 = \dfrac{669297}{538328}$ etc.

Helas, these recursions do not have a closed form.
No one has yet succeeded in expressing this Continued Fraction in known functions or known constants. Its decimal expansion is not recognized by inverse symbolic calculators.
So this monograph ends with an open ending.
The wait is for someone to bring light into the darkness with a clever recursion, an ingenious sequence or a marvelous integral…
To quote Jacob Bernoulli who wrote in 1689 after years of searching for the solution to a similar problem involving the summation of inverse squares:

"If anyone finds and communicates to us that which thus far has eluded our efforts, great will be our gratitude"

Appendix

Lemma 1: Euler's Continued Fraction formula

$$a_0 + a_0a_1 + a_0a_1a_2 + \ldots\ldots + a_0a_1a_2\ldots.a_n = \cfrac{a_0}{1 - \cfrac{a_1}{1+a_1 - \cfrac{a_2}{1+a_2 - \cfrac{a_3}{1+a_3 - \cfrac{a_4}{1+a_4 - \ddots}}}}}$$

This lemma is used in the proof of theorem 4.1, page 25.

Proof:

The sum of the first two consecutive terms can be expressed as

$$a_0 + a_0a_1 = a_0(1+a_1) = \cfrac{a_0}{\cfrac{1}{1+a_1}} = \cfrac{a_0}{1 - \cfrac{a_1}{1+a_1}} \quad = \cfrac{a_0}{1 - \cfrac{1}{1+\cfrac{1}{a_1}}} \qquad (1)$$

The sum of the first three consecutive terms can be expressed as

$$a_0 + a_0a_1 + a_0a_1a_2 = a_0(1+a_1(1+a_2))$$

Now we replace in (1) the term a_1 by $\cfrac{a_1}{1-\cfrac{a_2}{1+a_2}}$ to get

$$a_0 + a_0a_1 + a_0a_1a_2 = \cfrac{a_0}{1-\cfrac{1}{1+\cfrac{1}{a_1}}} = \cfrac{a_0}{1-\cfrac{1}{1+\cfrac{1-\cfrac{a_2}{1+a_2}}{a_1}}} = \cfrac{a_0}{1-\cfrac{a_1}{1+a_1-\cfrac{a_2}{1+a_2}}} = \cfrac{a_0}{1-\cfrac{a_1}{1+a_1-\cfrac{1}{1+\cfrac{1}{a_2}}}} \qquad (2)$$

For the sum of the first four consecutive terms is:
$a_0 + a_0a_1 + a_0a_1a_2 + a_0a_1a_2a_3 = a_0(1 + a_1(1 + a_2(1 + a_3)))$

We follow the same procedure.
By replacing in (2) the term a_2 by $\dfrac{a_2}{1 - \dfrac{a_3}{1 + a_3}}$ we get:

$$a_0 + a_0a_1 + a_0a_1a_2 + a_0a_1a_2a_3 = \cfrac{a_0}{1 - \cfrac{a_1}{1 + a_1 - \cfrac{1}{1 + \cfrac{1 - \cfrac{a_3}{1 + a_3}}{a_2}}}} = \cfrac{a_0}{1 - \cfrac{a_1}{1 + a_1 - \cfrac{a_2}{1 + a_2 - \cfrac{a_3}{1 + a_3}}}}$$

Obviously, this process can be continued indefinitely which proves lemma 1.

$$a_0 + a_0a_1 + a_0a_1a_2 + \ldots\ldots + a_0a_1a_2\ldots.a_n = \cfrac{a_0}{1 - \cfrac{a_1}{1 + a_1 - \cfrac{a_2}{1 + a_2 - \cfrac{a_3}{1 + a_3 - \cfrac{a_4}{1 + a_4 - \ddots}}}}}$$

Lemma 2: A variation of Euler's Continued Fraction formula

$$\frac{1}{b_0}-\frac{1}{b_1}+\frac{1}{b_2}-\frac{1}{b_3}+.....=\cfrac{1}{b_0+\cfrac{b_0^2}{b_1-b_0+\cfrac{b_1^2}{b_2-b_1+\cfrac{b_2^2}{b_3-b_2+\cfrac{b_3^2}{b_4-b_3+\ddots}}}}}$$

This lemma is used in the proof of theorem 4.1, page 25 and theorem 4.2, page 29.

Proof:

The summation $\frac{1}{b_0}-\frac{1}{b_1}+\frac{1}{b_2}-\frac{1}{b_3}+....$ can be rewritten as

$a_0+a_0a_1+a_0a_1a_2+......+a_0a_1a_2....a_n$ with

$a_0=\frac{1}{b_0}\quad a_1=-\frac{b_0}{b_1}\quad a_2=-\frac{b_1}{b_2}\quad a_3=-\frac{b_2}{b_3}$

Now applying Continued Fraction formula results in:

$$\frac{1}{b_0}-\frac{1}{b_1}+\frac{1}{b_2}-\frac{1}{b_3}+.....=\cfrac{1/b_0}{1+\cfrac{b_0/b_1}{1-b_0/b_1+\cfrac{b_1/b_2}{1-b_1/b_2+\cfrac{b_2/b_3}{1-b_2/b_3+\ddots}}}}=$$

Multiplying every numerator and denominator in the consecutive fractions with b_0,b_1,b_2 etc. proofs lemma 2.

$$\frac{1}{b_0}-\frac{1}{b_1}+\frac{1}{b_2}-\frac{1}{b_3}+.....=\cfrac{1}{b_0+\cfrac{b_0^2}{b_1-b_0+\cfrac{b_1^2}{b_2-b_1+\cfrac{b_2^2}{b_3-b_2+\cfrac{b_3^2}{b_4-b_3+\ddots}}}}}$$

Lemma 3: A Continued Fraction expressed as an integral nr. I

$$\int_0^1 \frac{x^{n-1}}{1+x^m}dx = \cfrac{1}{n+\cfrac{n^2}{m+\cfrac{(m+n)^2}{m+\cfrac{(2m+n)^2}{m+\cfrac{(3m+n)^2}{m+\ddots}}}}}$$

This lemma is used in the proof of theorem 4.1, page 25 and theorem 20, page 83.

Proof:

Consider the geometric series. $\frac{1}{1+x^m} = 1 - x^m + x^{2m} - x^{3m} + \dots$ and integrate.

$$\int_0^1 \frac{x^{n-1}}{1+x^m}dx = \int_0^1 x^{n-1} - x^{m+n-1} + x^{2m+n-1} - x^{3m+n-1} + \dots\, dx = \frac{1}{n} - \frac{1}{m+n} + \frac{1}{2m+n} - \frac{1}{3m+n}$$

The consecutive terms can be expressed as $a_0 + a_0a_1 + a_0a_1a_2 + a_0a_1a_2a_3$ when:

$a_0 = \frac{1}{n}$ $\quad a_1 = -\frac{n}{m+n}$ $\quad a_2 = -\frac{m+n}{2m+n}$ $\quad a_3 = -\frac{2m+n}{3m+n}$ $\quad a_4 = -\frac{3m+n}{4m+n}$ etc.

Applying Lemma 1 results in a Continued Fraction:

$$\int_0^1 \frac{x^{n-1}}{1+x^m}dx = \cfrac{\frac{1}{n}}{1+\cfrac{\frac{n}{m+n}}{1-\frac{n}{m+n}+\cfrac{\frac{m+n}{2m+n}}{1-\frac{m+n}{2m+n}+\cfrac{\frac{2m+n}{3m+n}}{1-\frac{2m+n}{3m+n}+\cfrac{\frac{3m+n}{4m+n}}{1-\frac{3m+n}{4m+n}-\ddots}}}}}$$

With a suitable multiplication this proofs lemma 3:

$$\int_0^1 \frac{x^{n-1}}{1+x^m}dx = \cfrac{1}{n+\cfrac{n^2}{m+\cfrac{(m+n)^2}{m+\cfrac{(2m+n)^2}{m+\cfrac{(3m+n)^2}{m+\ddots}}}}}$$

Lemma 4: Euler's Differential Method

This theorem is used in the proof of theorem 6, page 37, theorem 7, page 39, theorem 8 page 42 and 44, theorem 16, page 67, theorem 18, page 75, theorem 19, page 78 and in the proof of lemma 5, page 92, lemma 7, page 95 and lemma 8, page 97. These pages serve as a further explanation of Euler's method.

In his "*Observations*", 1739, L. Euler described an elegant way to convert certain Continued Fractions into integral expressions. He considered a sequence of integrals: $\int Pdx$, $\int PRdx$, $\int PR^2dx$, $\int PR^3dx$ ...., all with the property that these integrals vanish for $x=0$ and $x=1$ and with the property that $a\int Pdx = b\int PRdx + c\int PR^2dx$
Consequently, these integrals will also satisfy the following expression:

$$(a+\alpha n)\int_0^1 PR^n dx = (b+\beta n)\int_0^1 PR^{n+1}dx + (c+\gamma n)\int_0^1 PR^{n+2}dx$$

With $I_n = \int_0^1 PR^n dx$, this can be written as a recursion:

$$(a+\alpha n)I_n = (b+\beta n)I_{n+1} + (c+\gamma n)I_{n+2}$$

And this recursion can in turn be written as: $\dfrac{I_{n+1}}{I_n} = \dfrac{(a+\alpha n)}{(b+\beta n)+(c+\gamma n)\dfrac{I_{n+2}}{I_{n+1}}}$

Writing out a few terms shows the pattern of the Continued Fraction:

$$\frac{I_1}{I_0} = \cfrac{a}{b+\cfrac{(a+\alpha)\cdot c}{(b+\beta)+\cfrac{(a+2\alpha)\cdot(c+\gamma)}{(b+2\beta)+\cfrac{(a+3\alpha)\cdot(c+2\gamma)}{(b+3\beta)+\cfrac{(a+4\alpha)\cdot(c+3\gamma)}{(b+4\beta)+\cfrac{(a+5\alpha)\cdot(c+4\gamma)}{(b+5\beta)+\ddots}}}}}}$$

This pattern appears to occur in a large number of Continued Fractions.

Calculating $\dfrac{I_1}{I_0} = \dfrac{\int_0^1 PRdx}{\int_0^1 Pdx}$ will evaluate the corresponding Continued Fraction.

But $P(x)$ and $R(x)$ are still to be found.

The next brilliant idea of Euler was to search $P(x)$ and $R(x)$ as functions satisfying the corresponding identity with **in**definite integrals.
To achieve this, he introduced a new function $S(x)$.

$$(a+\alpha n)\int PR^{n}dx + R^{n+1}S = (b+\beta n)\int PR^{n+1}dx + (c+\gamma n)\int PR^{n+2}dx$$

As long as $R^{n+1}(x)\cdot S(x)=0$ for $x=0$ and $x=1$, this still represents a recursion and therefore a Continued Fraction.

The conversion towards *infinite* integrals allow to rewrite the expression with differentials: $(a+\alpha n)Pdx + (n+1)SdR + RdS = (b+\beta n)PRdx + (c+\gamma n)PR^{2}dx$
Since this must be valid for every *n*, it must be certainly true for $n=0$ and $n=1$

Therefore, $\begin{cases} aPdx + SdR + RdS = bPRdx + cPR^{2}dx \\ (a+\alpha)Pdx + 2SdR + RdS = (b+\beta)PRdx + (c+\gamma)PR^{2}dx \end{cases}$

Subtracting these two expressions results in

$$\begin{cases} aPdx + SdR + RdS = bPRdx + cPR^{2}dx \\ \alpha Pdx + SdR \qquad = \beta PRdx + \gamma PR^{2}dx \end{cases}$$

Solving for Pdx results in: $\begin{cases} SdR + RdS = \left(bR + cR^{2} - a\right)Pdx \\ SdR \qquad = \left(\beta Rdx + \gamma R^{2} - \alpha\right)Pdx \end{cases}$

and therefore in a differential equation in dS and dR

$$Pdx = \frac{SdR + RdS}{bR + cR^{2} - a} = \frac{SdR}{\beta R + \gamma R^{2} - \alpha}$$

Separating both variables results in the differential equation:

$$\frac{1}{S}dS = \frac{(b-\beta)R + (c-\gamma)R^{2} - (a-\alpha)}{\left(\beta R + \gamma R^{2} - \alpha\right)R}dR = \frac{(a-\alpha)}{\alpha R}dR + \frac{(\alpha b - a\beta) + (\alpha c - a\gamma)R}{\alpha\left(\beta R + \gamma R^{2} - \alpha\right)}dR$$

(The conversion to the second expression is a difficult one.)

After integrating both sides, a function $S = f(R)$ remains.
Now a function $R(x)$ must be sought, satisfying $R^{n+1}(x)\cdot S(x)$ for $x=0$ and $x=1$

If $R(x)$ exists, the Continued Fraction: $\dfrac{I_{n+1}}{I_n} = \dfrac{(a+\alpha n)}{(b+\beta n)+(c+\gamma n)\dfrac{I_{n+2}}{I_{n+1}}}$ is determined

by the integral quotient $\dfrac{I_1}{I_0} = \dfrac{\int_0^1 PRdx}{\int_0^1 Pdx} = \dfrac{\int_0^1 \frac{SR}{\beta R + \gamma R^{2} - \alpha}dR}{\int_0^1 \frac{S}{\beta R + \gamma R^{2} - \alpha}dR}$

Lemma 5: A Continued Fraction expressed as an integral nr. II

$$\frac{1}{s+\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{n^2}{s}\right)}=2\int_0^1 \frac{x^s}{1+x^2}dx$$

This lemma is used in the proof of theorem 7.1 – 7.5, page 39 and 40.

Proof:

Writing out a few terms, starting with w_1,shows that the recurrence relation

$w_n=\dfrac{n}{s+(1+n)w_{n+1}}$ equals the Continued Fraction $w_1=\cfrac{1^2}{s+\cfrac{2^2}{s+\cfrac{3^2}{s+\cfrac{4^2}{s+\cfrac{5^2}{s+\ddots}}}}}$

And therefore it follows $\dfrac{1}{w_1+s}=\dfrac{1}{s+\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{n^2}{s}\right)}$

Let $w_n=\dfrac{I_{n+1}}{I_n}$ which allows to rewrite the recurrence relation as $nI_n=sI_{n+1}+(1+n)I_{n+2}$

Choosing in lemma 4, page 90, $a=0,\ \alpha=1,\ b=s,\ \beta=0,\ c=1,\ \gamma=1$ results in the differential equation $\dfrac{1}{S}dS=\dfrac{s+R}{R^2-1}dR-\dfrac{1}{R}dR$ which has as its general solution

$\ln|S|=\ln\left|\dfrac{1}{R}\right|+\displaystyle\int\frac{s}{R^2-1}dR+\int\frac{R}{R^2-1}dR+c$ which can be written as:

$S=C\cdot\dfrac{1}{R}\left|\dfrac{R-1}{R+1}\right|^{\frac{s}{2}}\left|R^2-1\right|^{\frac{1}{2}}=C\cdot\dfrac{1}{R}\left(\dfrac{1-R}{1+R}\right)^{\frac{s}{2}}\cdot\sqrt{1-R^2}$ with $R<1$

Euler method involves the requirement that $R^{n+1}S=0$ for $x=0$ and for $x=1$ and this requirement is satisfied with the choice $R(x)=x$

Therefore $S(x)=C\cdot\dfrac{1}{x}\cdot\left(\dfrac{1-x}{1+x}\right)^{\frac{s}{2}}\cdot\sqrt{1-x^2}$

From $Pdx=\dfrac{SdR}{R^2-1}dx$ it follows that $Pdx=C^*\dfrac{1}{x}\cdot\left(\dfrac{1-x}{1+x}\right)^{\frac{s}{2}}\dfrac{1}{\sqrt{1-x^2}}dx$

Therefore, $w_1 = \frac{I_2}{I_1} = \frac{\int_0^1 PR^2 dx}{\int_0^1 PR dx} = \frac{\int_0^1 x \cdot \left(\frac{1-x}{1+x}\right)^{\frac{s}{2}} \frac{1}{\sqrt{1-x^2}} dx}{\int_0^1 \left(\frac{1-x}{1+x}\right)^{\frac{s}{2}} \frac{1}{\sqrt{1-x^2}} dx} = \frac{\text{Numerator}}{\text{Denominator}}$

This quotient of integrals can be evaluated as follows:

$$\text{Numerator} = \int_0^1 x\left(\frac{1-x}{1+x}\right)^{\frac{s}{2}} \cdot \frac{1}{\sqrt{1-x^2}} dx = -\int_0^1 \left(\frac{1-x}{1+x}\right)^{\frac{s}{2}} d\sqrt{1-x^2} = 1 + \int_0^1 \sqrt{1-x^2}\, d\left(\frac{1-x}{1+x}\right)^{\frac{s}{2}}$$

Let $A = \frac{1-x}{1+x} \Rightarrow x = \frac{1-A}{1+A} \Rightarrow \sqrt{1-x^2} = \frac{2\sqrt{A}}{1+A}$

$$\text{Numerator} = 1 + \int_0^1 \sqrt{1-x^2}\, d\left(\frac{1-x}{1+x}\right)^{\frac{s}{2}} = 1 + \int_1^0 \frac{2\sqrt{A}}{1+A} dA^{\frac{s}{2}} = 1 - s\int_0^1 \frac{\sqrt{A}\cdot A^{\frac{s}{2}-1}}{1+A} dA = 1 - s\int_0^1 \frac{A^{\frac{s-1}{2}}}{1+A} dA$$

The substitution $A = x^2$ results in $\text{Numerator} = 1 - s\int_0^1 \frac{A^{\frac{s-1}{2}}}{1+A} dA = 1 - 2s\int_0^1 \frac{x^s}{1+x^2} dx$

$$\text{Denominator} = \int_0^1 \left(\frac{1-x}{1+x}\right)^{\frac{s}{2}} \cdot \frac{1}{\sqrt{1-x^2}} dx = \int_1^0 A^{\frac{s}{2}} \cdot \frac{(1+A)}{2\sqrt{A}} d\frac{1-A}{1+A} = \int_0^1 \frac{A^{\frac{s-1}{2}}}{1+A} dA$$

The substitution $A = x^2$ results in $\text{Denominator} = 2\int_0^1 \frac{x^s}{1+x^2} dx$

Therefore: $w_1 = \frac{\int_0^1 PR^2 dx}{\int_0^1 PR dx} = \frac{\text{Numerator}}{\text{Denominator}} = \frac{1 - 2s\int_0^1 \frac{x^s}{1+x^2} dx}{2\int_0^1 \frac{x^s}{1+x^2} dx} = \frac{1}{2\int_0^1 \frac{x^s}{1+x^2} dx} - s$

Which can be rewritten as $\frac{1}{w_1 + s} = \cfrac{1}{s + \cfrac{1^2}{s + \cfrac{2^2}{s + \cfrac{3^2}{s + \ddots}}}} \Rightarrow \frac{1}{s + \underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{n^2}{s}\right)} = 2\int_0^1 \frac{x^s}{1+x^2} dx$

which proofs the lemma $\frac{1}{s + \underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{n^2}{s}\right)} = 2\int_0^1 \frac{x^s}{1+x^2} dx$

Lemma 6: Euler´s Reduction Method

$$\cfrac{1}{a+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{b+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{c+\ddots}}}}}}}=\cfrac{2}{2a+1+\cfrac{1}{2b+2+\cfrac{1}{2c+2+\ddots}}}$$

This lemma is used in the removal of "*repetitive ones*", theorem 11.1-11.4 on page 53 - 56.

Proof:

A basically simple but dedicated manipulation proofs this lemma.

$$A+\cfrac{1}{a+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{b+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{c+1}}}}}}}=A+\cfrac{1}{a+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{b+\cfrac{c+2}{2c+3}}}}}=$$

$$=A+\cfrac{2}{2a+\cfrac{8bc+8b+12c+14}{8bc+8b+8c+10}}=A+\cfrac{2}{2a+1+\cfrac{4c+4}{8bc+8b+8c+10}}=A+\cfrac{2}{2a+1+\cfrac{1}{\cfrac{8bc+8b+8c+10}{4c+4}}}=$$

$$=A+\cfrac{2}{2a+1+\cfrac{1}{2b+2+\cfrac{2}{4c+4}}}=A+\cfrac{2}{2a+1+\cfrac{1}{2b+2+\cfrac{1}{2c+2}}}$$

This procedure can be continued indefinitely which proves

$$A+\cfrac{1}{a+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{b+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{c+\ddots}}}}}}}=A+\cfrac{2}{2a+1+\cfrac{1}{2b+2+\cfrac{1}{2c+2+\ddots}}}$$

Lemma 7: A Continued Fraction expressed as an integral nr. III

$$s+\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{pn}{s+n}\right)=\frac{1}{e^{p}\int_{0}^{1}x^{s+p-1}e^{-px}dx}$$

This lemma is used in the proof of theorem 13.1 – 13.4, page 61.

Proof:

Let $w_n=\dfrac{p(1+n)}{s+1+n+w_{n+1}}$

Therefore, $w_0=\dfrac{p}{s+1+w_1}=\cfrac{p}{s+1+\cfrac{2p}{s+2+\cfrac{3p}{s+3+\cfrac{4p}{s+4+\ddots}}}}$

Now let $w_n=\dfrac{I_{n+1}}{I_n}$ in $w_n=\dfrac{p(1+n)}{s+1+n+w_{n+1}}$ This leads towards a recurrence relation.

$$\frac{I_{n+1}}{I_n}=\frac{p(1+n)}{s+1+n+\frac{I_{n+2}}{I_{n+1}}}\Rightarrow s+1+n+\frac{I_{n+2}}{I_{n+1}}=p(1+n)\frac{I_n}{I_{n+1}}\Rightarrow$$

$$p(1+n)I_n=(s+1+n)I_{n+1}+I_{n+2}$$

Applying '*Euler's Differential Method*' lemma 4, page 90 with
$a=p\quad \alpha=p\quad b=s+1\quad \beta=1\quad c=1\quad \gamma=0$ results in the differential equation:

$\dfrac{dS}{S}=\dfrac{sRdR+R^2dR}{R^2-pR}=\dfrac{s+R}{R-p}dR=dR+\dfrac{p+s}{R-p}dR$ which has as its general solution

$S=R+(p+s)\ln(R-P)+C \quad\Rightarrow\quad S=C\cdot e^{R}\cdot(R-p)^{p+s}$

Euler method involves the requirement that $R^{n+1}S=0$ for $x=0$ and for $x=1$ which is satisfied with the choice $R(x)=px$

Therefore $S=C\cdot e^{px}\cdot p^{p+s}(x-1)^{p+s}$

Now, I_0 and I_1 can be evaluated.

$$I_0=\int_0^1 Pdx \;=\int_0^1 \frac{S}{R-p}dR=\int_0^1 \frac{C\cdot e^{px}\cdot\left(px-p\right)^{p+s}}{px-p}\cdot pdx=C\cdot p^{p+s}\int_0^1 e^{px}\cdot\left(x-1\right)^{p+s-1}dx$$

$$I_1=\int_0^1 PRdx=\int_0^1 \frac{SR}{R-p}dR=\int_0^1 \frac{S\cdot(R-p)+pS}{R-p}dR \qquad =p\int_0^1 \frac{S}{R-p}dR+\int_0^1 SdR=pI_0+\int_0^1 SdR$$

$$I_1=pI_0+C\cdot p^{p+s}\int_0^1 e^{px}\left(x-1\right)^{p+s}pdx=pI_0+C\cdot p^{p+s+1}\int_0^1 e^{px}\left(x-1\right)^{p+s}dx$$

By partial integration it follows:

$$I_1=pI_0+C\cdot p^{p+s}\cdot(-1)^{p+s+1}-C\cdot p^{p+s}(p+s)\int_0^1 e^{px}\left(x-1\right)^{p+s-1}dx\Rightarrow$$

$$I_1=pI_0+C\cdot p^{p+s}\cdot(-1)^{p+s+1}-(p+s)\cdot I_0\;\Rightarrow I_1=-sI_0+C\cdot p^{p+s}\cdot(-1)^{p+s+1}$$

$$s+w_0=s+\frac{I_1}{I_0}=\frac{C\cdot p^{p+s}\cdot(-1)^{p+s+1}}{C\cdot p^{p+s}\int_0^1 e^{px}\cdot\left(x-1\right)^{p+s-1}dx}=\frac{(-1)^{p+s+1}}{(-1)^{p+s+1}\int_0^1 e^{px}\cdot\left(1-x\right)^{p+s-1}dx}\Rightarrow$$

$$s+\frac{I_1}{I_0}=\frac{1}{e^p\int_0^1 e^{-px}x^{p+s-1}dx}$$

Which completes the proof that

$$s+w_0=s+\frac{I_1}{I_0}=s+\operatorname*{K}_{n=1}^{\infty}\left(\frac{pn}{s+n}\right)=s+\cfrac{p}{s+1+\cfrac{2p}{s+2+\cfrac{3p}{s+3+\cfrac{4p}{s+4+\ddots}}}}=\frac{1}{e^p\int_0^1 e^{-px}x^{s+p-1}dx}$$

Lemma 8: Higher order roots

$$\left(1+p\right)\cdot\sqrt[p]{1+p}-1=1+2p+\underset{n=0}{\overset{\infty}{\mathrm{K}}}\left(\frac{(1+p)\cdot\frac{(1-(n+1)p)}{n+1}}{1+2p}\right)$$

This lemma is used in the proof of theorem 2.1 – 2.3, page 21.

First, as an illustration, we will proof theorem 2.1.

$$7+\cfrac{-\frac{2}{1}\cdot 4}{7+\cfrac{-\frac{5}{2}\cdot 4}{7+\cfrac{-\frac{8}{3}\cdot 4}{7+\cfrac{-\frac{11}{4}\cdot 4}{7+\cfrac{-\frac{14}{5}\cdot 4}{7+\ddots}}}}}=4\sqrt[3]{4}-1$$

Proof:

Let $w_0=\cfrac{1}{7+\cfrac{-\frac{2}{1}\cdot 4}{7+\cfrac{-\frac{5}{2}\cdot 4}{7+\cfrac{-\frac{8}{3}\cdot 4}{7+\cfrac{-\frac{11}{4}\cdot 4}{7+\cfrac{-\frac{14}{5}\cdot 4}{7+\ddots}}}}}}\quad\Rightarrow w_n=\dfrac{1}{7-4\cdot\frac{3n+2}{n+1}\cdot w_{n+1}}$

Let $w_n=\dfrac{I_{n+1}}{I_n}$ This results in the recursion $(1+n)I_n=(7+7n)I_{n+1}+(-8-12n)I_{n+2}$

Applying '*Euler's Differential Method*' lemma 4, page 90, results with $a=1\quad\alpha=1\quad b=7\quad\beta=7\quad c=-8\quad\gamma=-12$ in the differential equation

$\dfrac{dS}{S}=\dfrac{(-8+12)RdR}{-12R^2+7R-1}=\dfrac{-\frac{1}{3}RdR}{(R-\frac{1}{3})(R-\frac{1}{4})}=\left(\dfrac{-\frac{4}{3}}{R-\frac{1}{3}}+\dfrac{1}{R-\frac{1}{4}}\right)dR$ which has as its general solution $\ln(S)=\ln(\left|R-\tfrac{1}{4}\right|)-\tfrac{4}{3}\ln(\left|R-\tfrac{1}{3}\right|)\ \Rightarrow\ S=C\dfrac{\frac{1}{4}-R}{\left(\frac{1}{3}-R\right)^{\frac{4}{3}}}$

Euler method involves the requirement that $R^{n+1}S=0$ for $x=0$ and for $x=1$ and this requirement is satisfied when $R(x)=\frac{1}{4}x$

Which results in:

$w_0=\dfrac{I_1}{I_0}$ with $I_0=\displaystyle\int_0^1 Pdx=\int\frac{S}{\beta R+\gamma R^2-\alpha}dR$ and $I_1=\displaystyle\int_0^1 PRdx=\int\frac{SR}{\beta R+\gamma R^2-\alpha}dR$

Elaboration:

$$I_0 = \int \frac{S}{-12R^2+7R-1}dR = -\tfrac{1}{12}C\int \frac{\dfrac{\frac{1}{4}-R}{\left(\frac{1}{3}-R\right)^{\frac{4}{3}}}}{\left(\frac{1}{4}-R\right)\cdot\left(\frac{1}{3}-R\right)}dR = -\tfrac{1}{12}C\int_0^{\frac{1}{4}}\left(\tfrac{1}{3}-R\right)^{-\frac{7}{3}}dR =$$

$$= -\tfrac{1}{16}C\cdot 3\sqrt[3]{3}\cdot(4\sqrt[3]{4}-1)$$

$$I_1 = \int \frac{S\cdot R}{-12R^2+7R-1}dR = -\tfrac{1}{12}C\int_0^{\frac{1}{4}} \frac{\dfrac{\frac{1}{4}-R}{\left(\frac{1}{3}-R\right)^{\frac{4}{3}}}\cdot R}{\left(\frac{1}{4}-R\right)\cdot\left(\frac{1}{3}-R\right)}dR = -\tfrac{1}{12}C\int_0^{\frac{1}{4}}\frac{R}{\left(\frac{1}{3}-R\right)^{\frac{7}{3}}}dR =$$

$$I_1 = -\tfrac{1}{16}C\cdot 3\sqrt[3]{3}$$

(Which is easy to verify with symbolic calculators)

With these expressions for I_0 and I_1, the Continued Fraction can be evaluated as:

$$w_0 = \frac{I_1}{I_0} = \frac{-\frac{1}{16}C\cdot 3\sqrt[3]{3}}{-\frac{1}{16}C\cdot 3\sqrt[3]{3}\cdot(4\sqrt[3]{4}-1)} = \frac{1}{4\sqrt[3]{4}-1} = \cfrac{1}{7+\cfrac{-\frac{2}{1}\cdot 4}{7+\cfrac{-\frac{5}{2}\cdot 4}{7+\cfrac{-\frac{8}{3}\cdot 4}{7+\cfrac{-\frac{11}{4}\cdot 4}{7+\ddots}}}}}$$

Taking the reverse proofs theorem 2.1.

$$4\sqrt[3]{4}-1 = 7+\cfrac{-\frac{2}{1}\cdot 4}{7+\cfrac{-\frac{5}{2}\cdot 4}{7+\cfrac{-\frac{8}{3}\cdot 4}{7+\cfrac{-\frac{11}{4}\cdot 4}{7+\cfrac{-\frac{14}{5}\cdot 4\ddots}{7+\ddots}}}}} = 7+\operatorname*{K}_{n=0}^{\infty}\left(\frac{-4\cdot\frac{3n+2}{n+1}}{7}\right)$$

Theorem 2.2 can be proved in the same way.

The generalized Theorem 2.3 can also be proved in the same way, but the proof is more difficult.

$$(1+p)\sqrt[p]{p+1}-1=1+2p+\underset{n=0}{\overset{\infty}{\mathrm{K}}}\left(\frac{(1+p)\cdot\frac{(1-(n+1)p)}{n+1}}{1+2p}\right)$$

Proof:

This Continued Fraction can be written as:

$$w_0=\cfrac{1}{1+2p+\cfrac{\frac{(1-p)}{1}\cdot(1+p)}{1+2p+\cfrac{\frac{(1-2p)}{2}\cdot(1+p)}{1+2p+\cfrac{\frac{(1-3p)}{3}\cdot(1+p)}{1+2p+\cfrac{\frac{(1-4p)}{4}\cdot(1+p)}{1+2p+\ddots}}}}}$$ and this can be expressed as

the recurrence relation $w_n=\dfrac{1}{1+2p+(1+p)\cdot\frac{(1-(n+1)p)}{n+1}\cdot w_{n+1}}$

Let $w_n=\dfrac{I_{n+1}}{I_n}$ which allows o rewrite the recurrence as

$$(1+n)I_n=(1+2p)(1+n)I_{n+1}+\left(1-p^2-p(1+p)n\right)I_{n+2}$$

Applying '*Euler's Differential Method*' lemma 4, page 90, results with

$a=1\quad \alpha=1\quad b=1+2p\quad \beta=1+2p\quad c=1-p^2\quad \gamma=-p(1+p)$ in the differential equation $\dfrac{dS}{S}=\dfrac{(1-p^2+p(1+p)RdR}{(1+2p)R-p(1+p)R^2-1}=\dfrac{-1/pRdR}{(R-\frac{1}{p})(R-\frac{1}{p+1})}=\left(\dfrac{-1-\frac{1}{p}}{R-\frac{1}{p}}+\dfrac{1}{R-\frac{1}{p+1}}\right)dR$

A general solution is $S=C\dfrac{\frac{1}{p+1}-R}{\left(\frac{1}{p}-R\right)^{1+\frac{1}{p}}}$

Which results in:

$w_0=\dfrac{I_1}{I_0}$ with $I_0=\displaystyle\int_0^1 Pdx=\int\frac{S}{\beta R+\gamma R^2-\alpha}dR$ and $I_1=\displaystyle\int_0^1 PRdx=\int\frac{SR}{\beta R+\gamma R^2-\alpha}dR$

Elaboration:

First we will evaluate $I_0 = \int \frac{S}{\beta R + \gamma R^2 - \alpha} dR$

The denominator can be written as

$(1+2p)R - p(1+p)R^2 - 1 = -p(1+p)\cdot\left(\frac{1}{p+1} - R\right)\cdot\left(\frac{1}{p} - R\right)$

Since $S = C\frac{\frac{1}{p+1} - R}{\left(\frac{1}{p} - R\right)^{1+\frac{1}{p}}}$, I_0 can be written as

$$I_0 = -\frac{1}{p(1+p)} C \int \frac{\frac{\frac{1}{p+1} - R}{\left(\frac{1}{p} - R\right)^{1+\frac{1}{p}}}}{\left(\frac{1}{p+1} - R\right)\cdot\left(\frac{1}{p} - R\right)} dR = -\frac{1}{p(1+p)} C \int \left(\frac{1}{p} - R\right)^{-(2+\frac{1}{p})} dR$$

Euler method involves the requirement that $R^{n+1}S = 0$ for $x = 0$ and for $x = 1$ and this requirement is satisfied when $R(x) = \frac{1}{p+1}x$

But even without substituting $R(x) = \frac{1}{p+1}x$, the integral can be evaluated.

As the boundaries of integration we will use $R(0) = 0$ and $R(1) = \frac{1}{p+1}$

The integration requires some attention.

$$I_0 = -\frac{1}{p(1+p)}\cdot C\cdot\int_0^{\frac{1}{p+1}} \left(\frac{1}{p} - R\right)^{-(2+\frac{1}{p})} dR \qquad = \frac{1}{p(1+p)}\cdot C\cdot\int_0^{\frac{1}{p+1}} \left(\frac{1}{p} - R\right)^{-(2+\frac{1}{p})} d\left(\frac{1}{p} - R\right) =$$

$$= -\frac{1}{p(1+p)(1+\frac{1}{p})}\cdot C\cdot\left(\frac{1}{p} - R\right)^{-(1+\frac{1}{p})}\Bigg|_0^{\frac{1}{p+1}} \quad = -\frac{1}{(1+p)^2}\cdot C\cdot\left(\left(\frac{1}{p} - \frac{1}{p+1}\right)^{-(1+\frac{1}{p})} - \left(\frac{1}{p}\right)^{-(1+\frac{1}{p})}\right) =$$

$$= -\frac{1}{(1+p)^2}\cdot C\cdot\left(\left(p(1+p)\right)^{1+\frac{1}{p}} - p^{1+\frac{1}{p}}\right) = -\frac{1}{(1+p)^2}\cdot C\cdot p^{1+\frac{1}{p}}\left(\left(1+p\right)^{1+\frac{1}{p}} - 1\right) \Rightarrow$$

We have established that $I_0 = -\frac{1}{(1+p)^2}\cdot C\cdot p^{1+\frac{1}{p}}\left(\left(1+p\right)^{1+\frac{1}{p}} - 1\right)$

The evaluation of $I_1=\int\frac{S\cdot R}{(1+2p)R-p(1+p)R^2-1}dR$ is considerable more difficult and requires a lot of dedication.

$$I_1=\int\frac{S\cdot R}{(1+2p)R-p(1+p)R^2-1}dR=-\tfrac{1}{p(1+p)}\int\frac{S\cdot R}{R^2-\frac{1+2p}{p(1+p)}R+\frac{1}{p(1+p)}}dR=$$

$$-\tfrac{1}{p(1+p)}C\int_0^{\frac{1}{1+p}}\frac{R}{(\frac{1}{p}-R)^{2+\frac{1}{p}}}dR=\tfrac{1}{p(1+p)}C\int_0^{\frac{1}{1+p}}\frac{\frac{1}{p}-R}{\left(\frac{1}{p}-R\right)^{2+\frac{1}{p}}}dR-\tfrac{1}{p(1+p)}C\int_0^{\frac{1}{1+p}}\frac{\frac{1}{p}}{\left(\frac{1}{p}-R\right)^{2+\frac{1}{p}}}dR=$$

$$=-\tfrac{1}{p(1+p)}C\int_0^{\frac{1}{1+p}}\left(\tfrac{1}{p}-R\right)^{-(1+\frac{1}{p})}d\left(\tfrac{1}{p}-R\right)+\tfrac{\frac{1}{p}}{p(1+p)}C\int_0^{\frac{1}{1+p}}\left(\tfrac{1}{p}-R\right)^{-(2+\frac{1}{p})}d\left(\tfrac{1}{p}-R\right)=$$

$$=\tfrac{1}{p(1+p)\frac{1}{p}}C\left(\tfrac{1}{p}-R\right)^{-\frac{1}{p}}\Big|_0^{\frac{1}{p+1}}\quad-\quad\tfrac{\frac{1}{p}}{p(1+p)(1+\frac{1}{p})}C\left(\tfrac{1}{p}-R\right)^{-(1+\frac{1}{p})}\Big|_0^{\frac{1}{p+1}}=$$

$$=\tfrac{1}{1+p}C\left(\left(p(p+1)\right)^{\frac{1}{p}}-p^{\frac{1}{p}}\right)-\tfrac{\frac{1}{p}}{(1+p)^2}C\left(\left(p(p+1)\right)^{1+\frac{1}{p}}-p^{1+\frac{1}{p}}\right)=$$

$$=\tfrac{p^{\frac{1}{p}}}{(1+p)}C\left(\left(p+1\right)^{\frac{1}{p}}-1\right)-\tfrac{\frac{1}{p}\cdot p^{1+\frac{1}{p}}}{(1+p)^2}C\left(\left(p+1\right)^{1+\frac{1}{p}}-1\right)=$$

$$=\tfrac{p^{\frac{1}{p}}}{(1+p)}C\left(\left(p+1\right)^{\frac{1}{p}}-1\right)-\tfrac{p^{\frac{1}{p}}}{(1+p)^2}C\left(\left(p+1\right)^{1+\frac{1}{p}}-1\right)=$$

$$=\tfrac{p^{\frac{1}{p}}}{(1+p)}C\left(p+1\right)^{\frac{1}{p}}-\tfrac{p^{\frac{1}{p}}}{(1+p)}C-\tfrac{p^{\frac{1}{p}}}{(1+p)}C\left(p+1\right)^{\frac{1}{p}}+\tfrac{p^{\frac{1}{p}}}{(1+p)^2}C=$$

$$=-\tfrac{p^{1+\frac{1}{p}}}{(1+p)^2}C\qquad I_1=-C\tfrac{p^{1+\frac{1}{p}}}{(1+p)^2}$$

Now I_0 and I_1 are known.

$$w_0=\frac{I_1}{I_0}=\frac{I_1=-C\frac{p^{1+\frac{1}{p}}}{(1+p)^2}}{\frac{-1}{(1+p)^2}C\cdot p^{1+\frac{1}{p}}\left(\left(1+p\right)^{1+\frac{1}{p}}-1\right)}=\frac{1}{\left(\left(1+p\right)^{1+\frac{1}{p}}-1\right)}=\frac{1}{(1+p)\sqrt[p]{1+p}-1}$$

Therefore:

$$\frac{1}{w_0}=(1+p)\sqrt[p]{1+p}-1=1+2p+\cfrac{\frac{(1-p)}{1}\cdot(1+p)}{1+2p+\cfrac{\frac{(1-2p)}{2}\cdot(1+p)}{1+2p+\cfrac{\frac{(1-3p)}{3}\cdot(1+p)}{1+2p+\cfrac{\frac{(1-4p)}{4}\cdot(1+p)}{1+2p+\ddots}}}}\Rightarrow$$

Which proves that $(1+p)\sqrt[p]{p+1}-1=1+2p+\operatorname*{K}_{n=0}^{\infty}\left(\frac{(1+p)\cdot\frac{(1-(n+1)p)}{n+1}}{1+2p}\right)$

Lemma 9: Van de Veen's formula for higher order roots

$$\frac{\sqrt[n]{A}+1}{\sqrt[n]{A}-1}=1\cdot C+\cfrac{1-(1\cdot n)^2}{3\cdot C+\cfrac{1-(2\cdot n)^2}{5\cdot C+\cfrac{1-(3\cdot n)^2}{7\cdot C+\cfrac{1-(4\cdot n)^2}{9\cdot C+\cfrac{1-(5\cdot n)^2}{11\cdot C+\ddots}}}}} \quad \text{met } C=n\cdot\frac{A+1}{A-1}$$

This lemma is used on page 22, theorem 3.1 – 3.4, page 22.

Proof:

M. Sardina described in 2007 a general approach.
His idea is to write $r=\sqrt[n]{A}$ as $r=\sqrt[n]{\alpha^n+\beta}$ with α en β positive integers and to use $r=\alpha+\delta$ as an approximation. This turns out to a recurrence relation for δ which can be converted into a Continued Fraction.
An example with $\sqrt[3]{A}$ to demonstrate this approach:

$$r=\sqrt[3]{A}=\sqrt[3]{\alpha^3+\beta}=\alpha+\delta \;\Rightarrow\; \alpha^3+\beta=(\alpha+\delta)^3=\alpha^3+3\alpha^2\delta+3\alpha\delta^2+\delta^3$$

Therefore, δ can be calculated recursively. $\delta_k=\dfrac{\beta}{3\alpha^2+3\alpha\delta_{k-1}+\delta_{k-1}^2}$ and this can be written out as Continued Fraction. Assume $\delta_0=0$

Since $\delta_k=\dfrac{\beta}{3\alpha^2+3\alpha\delta_{k-1}+\delta_{k-1}^2}$ it follows that $\delta_1=\dfrac{\beta}{3\alpha^2}$ therefore $\delta_{1^e\text{ approximation}}=\dfrac{\beta}{3\alpha^2}$

$\delta_2=\dfrac{\beta}{3\alpha^2+\dfrac{\beta}{\alpha}+\dfrac{\beta^2}{9\alpha^4}}$ therefore $\delta_{2^e\text{ approximation}}=\dfrac{\beta}{3\alpha^2+\dfrac{\beta}{\alpha}}$

Now calculating δ_3 from δ_2 leads to

$$\delta_3=\frac{\beta}{3\alpha^2+\dfrac{3\alpha\beta}{3\alpha^2+\dfrac{\beta}{\alpha}+\dfrac{\beta^2}{9\alpha^4}}+\left(\dfrac{\beta}{3\alpha^2+\dfrac{\beta}{\alpha}+\dfrac{\beta^2}{9\alpha^4}}\right)^2}=\cfrac{\beta}{3\alpha^2+\cfrac{\beta}{\alpha+\cfrac{2\beta}{9\alpha^2}+\dots}}$$

Therefore $\delta_{3^e\text{ approximation}}=\cfrac{\beta}{3\alpha^2+\cfrac{\beta}{\alpha+\cfrac{2\beta}{9\alpha^2}}}$ (after intricate algebraic manipulations)

Continuation to next values requires a massive amount of manipulations...
But, at last a pattern emerge.

$$r=\sqrt[3]{\alpha^3+\beta}\ \Rightarrow r=\alpha+\cfrac{\beta}{3\alpha^2+\cfrac{2\beta}{2\alpha+\cfrac{4\beta}{9\alpha^2+\cfrac{5\beta}{2\alpha+\cfrac{7\beta}{15\alpha^2+\cfrac{8\beta}{2\alpha+\cfrac{10\beta}{21\alpha^2+\ddots}}}}}}}$$

In general: from $r=\sqrt[n]{\alpha^n+\beta}=\alpha+\delta$, it follows that

$\alpha^n+\beta=\alpha^n+\binom{n}{1}\alpha^{n-1}\delta+\binom{n}{2}\alpha^{n-2}\delta^2+....$ which leads to the recursive relation

$\delta_k=\dfrac{\beta}{\binom{n}{1}\alpha^{n-1}+\binom{n}{2}\alpha^{n-2}\delta_{k-1}+\binom{n}{3}\alpha^{n-3}\delta_{k-1}^2+...}$ and to the Continued Fraction:

$$r=\sqrt[n]{A}\ \Rightarrow r=\alpha+\cfrac{\beta}{n\alpha^{n-1}+\cfrac{\beta(n-1)}{2\alpha+\cfrac{\beta(n+1)}{3n\alpha^{n-1}+\cfrac{\beta(2n-1)}{2\alpha+\cfrac{\beta(2n+1)}{5n\alpha^{n-1}+\cfrac{\beta(3n-1)}{2\alpha+\cfrac{\beta(3n+1)}{7n\alpha^{n-1}+\ddots}}}}}}}$$

The repetitive occurrence of the factor 2α can be removed by combining two terms, similar to the Euler's Reduction Method.
Applying this reduction results in the more general Sardina Theorem:

$$r=\sqrt[n]{A}=\left(\alpha^n+\beta\right)^{\frac{1}{n}}=$$

$$=\alpha+\cfrac{2\alpha\beta}{2nA-\beta(1+n)-\cfrac{\left((1n)^2-1\right)\beta^2}{3n(2A-\beta)-\cfrac{\left((2n)^2-1\right)\beta^2}{5n(2A-\beta)-\cfrac{\left((3n)^2-1\right)\beta^2}{7n(2A-\beta)-\cfrac{\left((4n)^2-1\right)\beta^2}{9n(2A-\beta)-\ddots}}}}}$$

Multiple division by β and adjusting for the first 'irregular' term results after some small algebraic manipulations in the expression:

$$\frac{\sqrt[n]{A}+\alpha}{\sqrt[n]{A}-\alpha}=1\cdot n(2\tfrac{A}{\beta}-1)+\cfrac{1-(1n)^2}{3\cdot n(2\tfrac{A}{\beta}-1)+\cfrac{1-(2n)^2}{5\cdot n(2\tfrac{A}{\beta}-1)+\cfrac{1-(3n)^2}{7\cdot n(2\tfrac{A}{\beta}-1)+\cfrac{1-(4n)^2}{9\cdot n(2\tfrac{A}{\beta}-1)+\ddots}}}}$$

Because $\alpha^n+\beta=A$, the repeating term $2\tfrac{A}{\beta}-1$ can be rewritten as:

$$2\tfrac{A}{\beta}-1=2\frac{A}{A-\alpha^n}-1=\frac{2A}{A-\alpha^n}-\frac{A-\alpha^n}{A-\alpha^n}=\frac{A+\alpha^n}{A-\alpha^n}$$

After the obvious substitution $C=n\cdot\dfrac{A+\alpha^n}{A-\alpha^n}$ the following theorem results:

$$\frac{\sqrt[n]{A}+\alpha}{\sqrt[n]{A}-\alpha}=1\cdot C+\cfrac{1-(1\cdot n)^2}{3\cdot C+\cfrac{1-(2\cdot n)^2}{5\cdot C+\cfrac{1-(3\cdot n)^2}{7\cdot C+\cfrac{1-(4\cdot n)^2}{9\cdot C+\ddots}}}}$$

The choice $\alpha=1$ results in an elegant Continued Fraction not previously found in the literature: "*Van de Veen's formula for higher order roots*".

$$\frac{\sqrt[n]{A}+1}{\sqrt[n]{A}-1}=1\cdot C+\cfrac{1-(1\cdot n)^2}{3\cdot C+\cfrac{1-(2\cdot n)^2}{5\cdot C+\cfrac{1-(3\cdot n)^2}{7\cdot C+\cfrac{1-(4\cdot n)^2}{9\cdot C+\ddots}}}}\quad \text{met}\quad C=n\cdot\frac{A+1}{A-1}$$

Choosing a specific *n* and *A* determines *C* and the Continued Fraction for $\sqrt[n]{A}$

When we define a "*nice-looking*" Continued Fraction as containing only integers in its expansion, *C* must be an integer. But since $C = n \cdot \dfrac{A+1}{A-1}$, this restricts n and A

There are five possibilities resulting in an integer *C*:

I. $A=2 \quad \Rightarrow C=3n$ the group of Continued Fractions for $\sqrt[n]{2}$

An example is theorem 3.1: $n=3, \quad A=2, \quad C=3\cdot\frac{2+1}{1}=9$

$$\frac{\sqrt[3]{2}+1}{\sqrt[3]{2}-1}=1\cdot 9+\cfrac{1-3^2}{3\cdot 9+\cfrac{1-6^2}{5\cdot 9+\cfrac{1-9^2}{7\cdot 9+\cfrac{1-12^2}{9\cdot 9+\ddots}}}}$$

II. $A=3 \quad \Rightarrow C=2n$ the group of Continued Fractions for $\sqrt[n]{3}$

An example is theorem 3.2: $n=3, \quad A=3, \quad C=3\cdot\frac{3+1}{2}=6$

$$\frac{\sqrt[3]{3}+1}{\sqrt[3]{3}-1}=1\cdot 6+\cfrac{1-3^2}{3\cdot 6+\cfrac{1-6^2}{5\cdot 6+\cfrac{1-9^2}{7\cdot 6+\cfrac{1-12^2}{9\cdot 6+\ddots}}}}$$

III. $A=n+1 \quad \Rightarrow C=n+2$ the group of Continued Fractions for $\sqrt[n]{n+1}$

An example is theorem 3.3: $n=3, \quad A=4, \quad C=5$

$$\frac{\sqrt[3]{4}+1}{\sqrt[3]{4}-1}=1\cdot 5+\cfrac{1-3^2}{3\cdot 5+\cfrac{1-6^2}{5\cdot 5+\cfrac{1-9^2}{7\cdot 5+\cfrac{1-12^2}{9\cdot 5+\ddots}}}}$$

IV. $A=2n+1 \Rightarrow C=n+1$ the group of Continued Fractions for $\sqrt[n]{2n+1}$

An example is theorem 3.4: $n=7, \quad A=15, \quad C=7\cdot\frac{15+1}{14}=8$

$$\frac{\sqrt[7]{15}+1}{\sqrt[7]{15}-1}=1\cdot 8+\cfrac{1-7^2}{3\cdot 8+\cfrac{1-14^2}{5\cdot 8+\cfrac{1-21^2}{7\cdot 8+\cfrac{1-28^2}{9\cdot 8+\ddots}}}}$$

V. $A=1+\dfrac{n}{k} \quad \Rightarrow C=k\left(A+1\right)$ the group of Continued Fractions for $\sqrt[n]{\frac{n}{k}+1}$

Which results from assuming $n=k\cdot(A-1)$ with $k\geq 2$

An example is theorem 3.5: $k=2, \ n=8 \ \Rightarrow A=5$ and $C=12$ results in the following but inelegant Continued Fraction for $\sqrt[8]{5}$

$$\frac{\sqrt[8]{5}+1}{\sqrt[8]{5}-1}=1\cdot 12+\cfrac{1-8^2}{3\cdot 12+\cfrac{1-16^2}{5\cdot 12+\cfrac{1-24^2}{7\cdot 12+\cfrac{1-32^2}{9\cdot 12+\ddots}}}}$$

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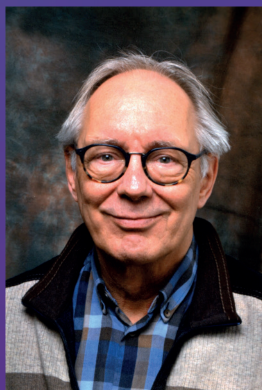
Some Continued Fractions and proofs

Mathematics is science and art as well. It is the never-ending search for structure, regularity and patterns between numbers, shapes and concepts. These patterns already exist and are patiently waiting to be discovered. They often show a timeless beauty, regardless of any practical application.
Continued Facturions are a typical example. They are useless for calculations in today's computer age. They possess intrinsic aesthetic properties, along with a deceptive simplicity, using nothing else than the elementary operations of addition, subtraction, multiplication and division. They have attracted the attention of many over the centuries. Like Lord Brouncker (1620-1684), who discovered a beautiful infinite continued fraction for the magic constant π. Later Euler contributed with many other continued fractions in his dissertation "*De fractionibus continuis*".

This book contains a collection of many well-known continued fractions and a few surprising new ones. It also contains proofs because mathematics should offer inextricable evidence for observed patterns and regularity. Many of these proofs were known in the 17th and 18th centuries but have long been hidden in hard-to-access texts. Besides revived proofs in modern notation, this book also contains a couple of new and original proofs.

ABOUT THE AUTHOR



The author has practised several diverse disciplines.
To name a few, engineering and designing of prosthetic limbs, playing draughts, performing as a magician and contortionist, an expert in the construction of polytopes as the diprismatohexacosihecatonicosachoron and finally, mathematics.
It shows his curiosity and his fascination for the exotic.
It was only a matter of time before Continued Fractions crossed his path. Maybe it is more appropriate to state that Continued Fractions have chosen the author, in the hope to show their magic on stage again and return to their former glory.

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