The Story of the 120-Cell

John Stillwell

One of the most beautiful objects in mathematics is the regular polytope in $\mathbb{R}^4$ whose boundary consists of 120 dodecahedral cells. This 120-cell is a rarity among rarities because it lives in three very special worlds. Its home is among the regular polytopes in $\mathbb{R}^4$, but it also lives in the remarkable sphere $\mathbb{S}^3$ and in the quaternions $\mathbb{H}$. And if this is not enough, the 120-cell encodes the symmetry of the icosahedron and the structure of the Poincaré homology sphere. All these facts have been known since the 1930s, but the story can be told more elegantly in contemporary language, and it can be *illustrated* better than ever before with the help of computer graphics. Moreover, the new illustrations put the 120-cell in a context of current interest, the geometry of soap bubble configurations, by mapping it in a natural way into $\mathbb{R}^3$ (cover illustration).

Telling the story in contemporary language has the danger that certain connections become "obvious", and it is hard to understand how our mathematical ancestors could have overlooked them. However, there is no turning back; we cannot stop seeing the connections we see now, so the best thing to do is describe them as clearly as possible and recognise that our ancestors lacked our advantages.

The story begins with the first encounters with the fourth dimension in the 1840s, becomes entangled with group theory in the 1850s, and interacts with topology around 1900. But to set the scene properly, we should review the regular polyhedra, because they are the origin of everything we are going to discuss.

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The author received support from the Dibner Institute at MIT while writing this article, which is part of a forthcoming larger project on exotic symmetry in algebra and geometry.

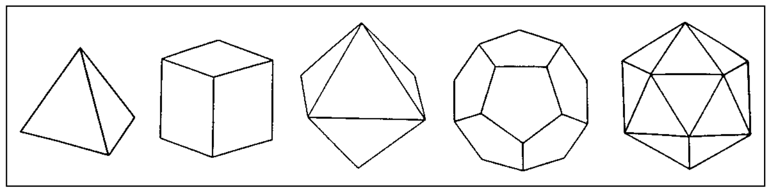

Figure 1. The five regular polyhedra.

The Regular Polyhedra

The five regular polyhedra existed before human history (for example, in the form of crystals and viruses), and they certainly made an early appearance in the history of mathematics. They are the climax of Euclid's *Elements*.

There are several proofs that these five polyhedra are the only regular ones: the classical proof enumerating which polygons can occur as faces and which angle sums are possible at a vertex, the topological proof showing that everything is controlled by the Euler characteristic, and the nice spherical geometry proof of Legendre. A less elegant proof, but one that generalises to higher dimensions, considers the ratio of edge length to the diameter of the circumscribed sphere and relates it to the corresponding ratio of the lower-dimensional *vertex figure* (the convex hull of the neighbouring vertices of a given vertex).

Call this ratio ER. It is easy to prove that if Π is a polyhedron with p-gons as faces and if the polygon Π' is its vertex figure, then

$$\mathrm{ER}(\Pi)^2 = 1 - \frac{\cos^2 \frac{\pi}{p}}{\mathrm{ER}(\Pi')^2}.$$

In particular, we must have $1 > \cos\frac{\pi}{p}/\mathrm{ER}(\Pi')$; and when Π' is a regular q-gon, then $\mathrm{ER}(\Pi') = \sin\frac{\pi}{q}$, so

$$\cos\frac{\pi}{p} < \sin\frac{\pi}{q}.$$

The only pairs (p, q) of edge numbers satisfying this condition are those corresponding to the five standard polyhedra.

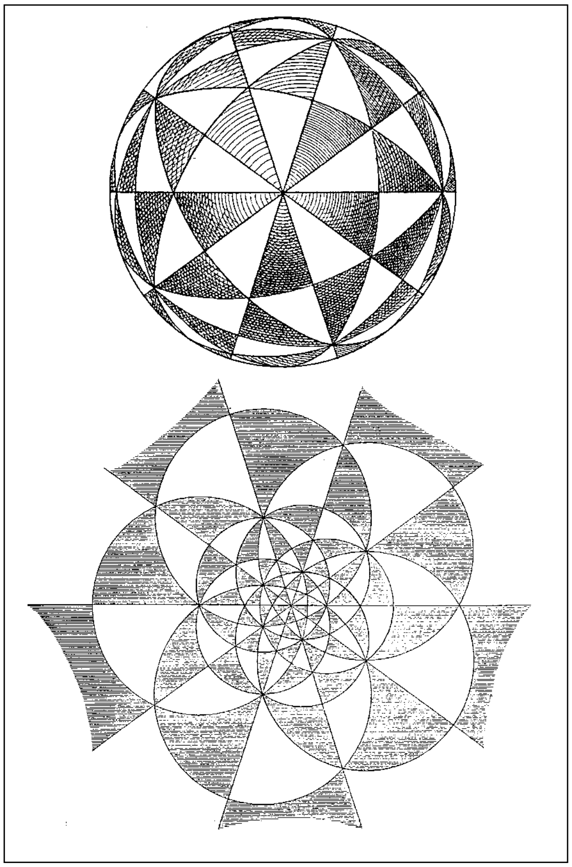

Figure 2. Icosahedral tilings of $\mathbb{S}^2$ and $\mathbb{R}^2$.

The advantage of this proof is the *dimension reduction* from Π to Π'. A generalisation of the proof allows the regular figures Π in $\mathbb{R}^{n+1}$ to be derived from the known regular figures in $\mathbb{R}^n$.

A different dimension reduction, useful for purposes of visualisation, can also be illustrated with the regular polyhedra. The vertices of a regular polyhedron Π lie on the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$, so projecting Π from its centre onto $\mathbb{S}^2$ produces a regular tiling of $\mathbb{S}^2$ by spherical polygons. This tiling can in turn be mapped onto $\mathbb{R}^2$ by stereographic projection. Felix Klein used this idea to visualise the symmetry groups of the regular polyhedra, and some magnificent pictures of the relevant tilings were produced at his instigation. Two of them are shown in Figure 2, which is from Klein and Fricke [10, pp. 105–106]. These tilings result from the icosahedron (and its dual dodecahedron) when each face is subdivided by its axes of symmetry.

The same idea applies to regular polytopes in $\mathbb{R}^4$ and is particularly useful for visualising the 120-cell. Projecting the 120-cell onto $\mathbb{S}^3$ and then onto $\mathbb{R}^3$ gives images that are visible and mathematically significant. In particular, we shall find that the 120-cell gives another view of the rotation group of the icosahedron.

Klein discovered that the rotation groups of the tetrahedron, cube (and its dual octahedron), and icosahedron (and its dual dodecahedron) are none other than the alternating and symmetric groups A_4, S_4, and A_5 respectively. In his famous *Lectures on the Icosahedron*, Klein made a meal of the resulting connection between the regular polyhedra and the solution of quintic equations.

But as early as 1856, long before Klein's work, William Rowan Hamilton found the algebraic equivalent of Figure 2: a *presentation* of the icosahedral group by generators and relations. This was the first significant result in combinatorial group theory. In a follow-up paper [8, p. 609] Hamilton presented all three polyhedral groups: the group $\mathcal{T}$ of rotations of the tetrahedron, the group $\mathcal{O}$ of rotations of the octahedron (and cube), and the group $\mathcal{I}$ of rotations of the icosahedron (and dodecahedron).

Hamilton defined $\mathcal{I}$ by three symbols ι, κ, and λ, and the relations

$$\iota^2 = \kappa^3 = \lambda^5 = 1, \quad \lambda = \iota\kappa.$$

Since λ is redundant, the group may be defined more concisely by

$$\iota^2 = \kappa^3 = (\iota\kappa)^5 = 1.$$

The symbol ι can be interpreted as a half turn of an icosahedron about an axis through the midpoints of opposite edges. (In Figure 2 such a midpoint corresponds to a vertex where two black and two white triangles meet.) The symbol κ represents a 1/3 turn about an axis through the midpoints of opposite faces (in Figure 2, vertices where three black and three white triangles meet).

The tiling in Figure 2 is thus a picture of Hamilton's presentation of $\mathcal{I}$, and of course it fits the underlying icosahedron/dodecahedron like a glove. Nevertheless, one might wonder whether there is a more *homogeneous* picture of $\mathcal{I}$, one in which there is only one type of vertex instead of three. Later we shall see that such a picture exists in three dimensions, and it is essentially the 120-cell.

Hamilton had a combinatorial interpretation of ι, κ, and λ as rules for "passage from face to face" on the polyhedron and may not have seen them as rotations. What is more surprising, coming from the inventor of the quaternion algebra, is that he did *not* regard ι, κ, and λ as quaternions. Today we are inclined to think that quaternions are the perfect way to represent rotations, and Hamilton is supposed to be the originator of this idea!

Quaternions

The quaternion algebra $\mathbb{H}$ belongs to the exclusive family of algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, which are the only *normed algebras* over $\mathbb{R}$—algebras for which the product uv of any elements u and v satisfies $|uv| = |u||v|$. Of the four algebras, $\mathbb{H}$ is probably the most fascinating. The complex field $\mathbb{C}$ is old hat to us now, and the octonion algebra $\mathbb{O}$ can be viewed (a little unfairly, but it is outside the scope of this article) as a spinoff of $\mathbb{H}$. The story of $\mathbb{H}$ is of course the story of Hamilton, but it is also the story of several other mathematicians, some of whom "sighted" aspects of quaternions earlier but did not fully understand what they had seen.

Hamilton found the quaternions while searching for normed algebras over $\mathbb{R}$ of arbitrary dimension. He observed in 1835 that $\mathbb{C}$ could be defined abstractly as $\mathbb{R}^2$ with the usual vector sum and with the far less obvious product defined by

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 - b_1b_2, b_1a_2 + a_1b_2).$$

This formula is inspired by hindsight, of course. It is precisely the definition we get by interpreting each complex number $a + ib$ as the ordered pair (a, b) of reals and working out which pair corresponds to $(a_1 + ib_1)(a_2 + ib_2)$ when i^2 is assumed to equal -1. But it *could* have come from insight: a sufficiently clever mathematician could have seen it coming from the *two-square identity*. This identity says that sums of squares are "multiplicative", in the sense that a number which factorises into sums of two squares,

$$(a_1^2 + b_1^2)(a_2^2 + b_2^2),$$

is itself a sum of two squares, namely,

$$(a_1a_2 - b_1b_2)^2 + (b_1a_2 + a_1b_2)^2.$$

This identity was probably known to Diophantus and was explicitly mentioned by later commentators on his work, such as Fibonacci in his 1225 *Book of Squares*. Today we would say that the identity expresses the multiplicative property of the *norm* $a^2 + b^2$ of the pair (a, b). (Diophantus's understanding was similar, though more concrete: (a, b) was the right-angled triangle with sides a and b, while $a^2 + b^2$ was the square on the hypotenuse.) This is the property that makes $\mathbb{C}$ a normed algebra, and it is the property that Hamilton hoped to find in higher dimensions.

His first attempt was to find a product formula for triples $(a, b, c) = a + ib + jc$ that would make $\mathbb{R}^3$ a normed algebra. Hamilton worked from about 1830 to 1843 on this problem, surprisingly unaware of known obstructions to its solution. The product formula on $\mathbb{R}^2 = \mathbb{C}$ implies, as we have seen, that sums of two squares are multiplicative. For example—and this is precisely the example of Diophantus from which later commentators inferred the two-square identity—

$$65 = 13 \cdot 5 = (3^2 + 2^2)(2^2 + 1^2),$$

and therefore

$$65 = (3 \cdot 2 - 2 \cdot 1)^2 + (2 \cdot 2 + 3 \cdot 1)^2 = 4^2 + 7^2.$$

A similar product formula with multiplicative norm on $\mathbb{R}^3$ would imply that an integer which factorises into sums of three squares is itself a sum of three squares, but this is simply not true! Consider

$$15 = 5 \cdot 3 = (2^2 + 1^2 + 0^2)(1^2 + 1^2 + 1^2),$$

for example. Another example, $63 = 21 \cdot 3$ (which uses only positive squares), had been published by Legendre, but Hamilton didn't come across it until after he had given up on $\mathbb{R}^3$.

He abandoned triples on 16 October 1843, when he saw that he needed not two imaginary units i and j but three: i, j, and $ij = k$ with

$$i^2 = j^2 = k^2 = ijk = -1.$$

These relations imply that $ij = -ji$, so the product is not commutative, but Hamilton was prepared for this possibility. He had already entertained it in his unsuccessful search for a product of triples. The product of quadruples has the saving grace of a multiplicative norm, so Hamilton had achieved his main goal: he had found a normed algebra of quadruples, now known as the *quaternion algebra* $\mathbb{H}$.

A consequence of the multiplicative norm is a *four-square identity*, which Hamilton at first thought was his own discovery. In fact, Euler knew it in 1748, and Lagrange used it in his well-known (to some mathematicians!) proof that every positive integer is the sum of four squares. Moreover, a remarkable restatement of the four-square identity as a *complex two-square identity* occurs in the unpublished work of Gauss (*Werke*, vol. 3, 383–4):

$$(|a_1|^2 + |b_1|^2)(|a_2|^2 + |b_2|^2) = |a_1a_2 - b_1\overline{b_2}|^2 + |b_1\overline{a_2} + a_1b_2|^2.$$

With its amazing similarity to the Diophantus identity, the Gauss identity prompts a definition of quaternion multiplication (equivalent to Hamilton's) as a product of pairs (a, b) of *complex* numbers,

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 - b_1\overline{b_2}, b_1\overline{a_2} + a_1b_2).$$

The complex two-square identity is related to another formula found by Gauss around 1819 (and discussed below), so perhaps it dates from the same period. With hindsight these results of Euler and Gauss may be regarded as "sightings" of $\mathbb{H}$, analogous to Diophantus's sighting of $\mathbb{C}$. Another sighting of $\mathbb{H}$, by Rodrigues in 1840, will be described when we discuss how quaternions are related to rotations. Of course, Hamilton's priority is secure; the merit of his discovery is only increased by mathematicians as great as Euler and Gauss having missed it, despite the glimpses they had caught.

Rotations

Hamilton was certain, as soon as he discovered the quaternions, that they would be worth studying for the rest of his life. His friends were not so sure. On 26 October 1843, John Graves, who had worked with Hamilton for years in the search for normed algebras, wrote:

> You must have been in a very bold mood to start the happy idea that ij might be different from ji …. Have you any inkling of the existence in nature of processes, or operations, or phenomena, or conceptions analogous to the circuit
>
> $$ij = -ji = k$$
> $$jk = -kj = i$$
> $$ki = -ik = j?$$

Hamilton replied with a hint of applications to physics and announced that quaternions could certainly be used to derive theorems of spherical trigonometry, but Graves was not satisfied:

> There is still something in the system that gravels me. I have not yet any clear views as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties….But supposing that your symbols have their physical antetypes, which might have led to your quaternions, what right have you to such luck, getting at your system by such an *inventive* mode as yours?

(For more of these letters, see [7, vol. 3, p. 443].)

Of course, Graves's remark about luck was tongue-in-cheek, and he got into the act himself in December 1843 by discovering an eight-square identity and with it the normed algebra $\mathbb{O}$ of octonions—but that is another story. The answer to his question was that "physical antetypes" of the quaternions had already been sighted twice, by Gauss and by Rodrigues. They are rotations in $\mathbb{R}^3$.

Again, there was a precedent in the history of complex numbers, which involves rotations in $\mathbb{R}^2$. Around 1590 Diophantus's process of generating the side pair $(a_1a_2 - b_1b_2, b_1a_2 + a_1b_2)$ of a right-angled triangle from side pairs (a_1, b_1) and (a_2, b_2) was found to hold a second secret. Viète showed, in his *Genesis triangulorum*, that the process not only multiplies the hypotenuses, it also *adds* the angles (between the first side and the hypotenuse). He had sighted the relation between angles and complex numbers that later took the shape of de Moivre's theorem.

The Gauss process, which takes pairs (a_1, b_1) and (a_2, b_2) of complex numbers and forms the pair $(a_1a_2 - b_1\overline{b_2}, b_1\overline{a_2} + a_1b_2)$, likewise combines rotations in $\mathbb{R}^3$. Gauss discovered this around 1819 (*Werke*, vol. 8, 354–362) by stereographically projecting the sphere $\mathbb{S}^2$ onto the plane. Interpreting the plane as $\mathbb{C}$, he found that each rotation of $\mathbb{S}^2$ induces a map of $\mathbb{C}$ of the form

$$z \mapsto \frac{az + b}{-\bar{b}z + \bar{a}}.$$

Thus each rotation of $\mathbb{S}^2$ (hence of $\mathbb{R}^3$) may be parametrised by a pair (a, b) of complex numbers. When the rotations parametrised by the pairs (a_1, b_1) and (a_2, b_2) are combined, the resulting rotation is the one parametrised by $(a_1a_2 - b_1\overline{b_2}, b_1\overline{a_2} + a_1b_2)$.

Gauss did not publish this result, however, and it was rediscovered in 1879 by Cayley [1, vol. X, p. 153]. It leads to an elegant matrix representation of $\mathbb{H}$, already hinted at in Cayley's 1858 paper on matrices [1, vol. II, p. 491]. Instead of the linear fractional function $z \mapsto \dfrac{az + b}{-\bar{b}z + \bar{a}}$, one takes the matrix

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

which can be decomposed into a sum

$$\alpha\mathbf{1} + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}$$

by setting $a = \alpha + i\beta$ and $b = \gamma + i\delta$, where $\alpha, \beta, \gamma, \delta$ are real. The matrices $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

respectively, and they satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}.$$

Thus Cayley's matrices $\alpha\mathbf{1} + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}$ correspond isomorphically to the quaternions $\alpha + i\beta + j\gamma + k\delta$.

Passing from the linear fractional function $\dfrac{az + b}{-\bar{b}z + \bar{a}}$ to the quaternion $\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right)$ introduces some ambiguity as far as rotations are concerned. Infinitely many matrices correspond to the same function, and even if we restrict to quaternions with norm $|a|^2 + |b|^2 = 1$, each rotation corresponds to a pair $\pm\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right)$. This cannot be helped, because the quaternions of norm 1 form an $\mathbb{S}^3$, whereas the rotations of $\mathbb{S}^2$ form the projective space $\mathbb{R}P^3$, which is not homeomorphic to $\mathbb{S}^3$. Indeed, $\mathbb{R}P^3$ is the space of antipodal point pairs of $\mathbb{S}^3$, which are precisely the matrix pairs $\pm\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right)$.

Since two quaternions correspond to each rotation, the polyhedral groups $\mathcal{T}$, $\mathcal{O}$, and $\mathcal{I}$ "lift" to quaternion groups of twice their respective sizes: the *binary tetrahedral group* $\mathbf{T}$ of 24 elements, the *binary octahedral group* $\mathbf{O}$ of 48 elements, and the *binary icosahedral group* $\mathbf{I}$ of 120 elements.

One might ask how are the four real parameters $\alpha, \beta, \gamma, \delta$ in the quaternion pair $\pm(\alpha + i\beta + j\gamma + k\delta)$ related to geometric parameters of the corresponding rotation? The answer is remarkably simple:

$$\alpha = \cos\frac{\theta}{2},\ \beta = \lambda\sin\frac{\theta}{2},\ \gamma = \mu\sin\frac{\theta}{2},\ \delta = \nu\sin\frac{\theta}{2},$$

where (λ, μ, ν) is a unit vector in the direction of the rotation axis and θ is the amount of rotation about the axis. This was discovered independently by Hamilton and Cayley in 1845 (see [1, vol. I, p. 123]). Cayley (as usual, better read than Hamilton) also noticed that the same parameters had already been used by Rodrigues in 1840, with a rule for computing the parameters of a composite rotation that was essentially the quaternion multiplication rule. Later, when Gauss's unpublished work came to light, it was found that he had the rule too.

In January 1863 Cayley [1, vol. V, p. 539] used the geometric parameters to write down the members of the binary polyhedral groups **T**, **O**, and **I** as quaternions. They form remarkably symmetrical sets in $\mathbb{R}^4$, and—unknown to Cayley—they had already been observed in geometry.

The Regular Polytopes

The quaternions were part of a wave of geometry in higher dimensions that welled up, rather mysteriously, in 1843–1844. Until that time, geometry in more than three dimensions had been even less welcome than non-Euclidean geometry. Yet in 1843, independently of Hamilton, Cayley published a paper entitled "Chapters in the analytical geometry of (*n*) dimensions", and Grassmann was writing the first edition of his book *Ausdehnungslehre*, published in 1844. Admittedly, the latter works barely registered in the general mathematical consciousness, but perhaps a mathematical unconsciousness was at work. By 1852 another mathematician working in obscurity had answered all the basic questions about higher-dimensional regular polyhedra, or *polytopes* as they are now called.

Ludwig Schläfli obtained these results, and much more, in his *Theorie der vielfachen Kontinuität* [13], written in 1852 but not published in its entirety until 1901, six years after his death. His results on polytopes did not become generally known until others rediscovered them in the 1880s.

Schläfli's main results were obtained using the ER formula mentioned in the section on polyhedra and may be expressed in current terminology as follows.

- In each $\mathbb{R}^n$ there are generalisations of the tetrahedron, the cube, and the octahedron, called the *n*-simplex, *n*-cube, and *n*-orthoplex respectively.
  The only regular polytopes other than these are the dodecahedron and icosahedron in $\mathbb{R}^3$ and three special polytopes in $\mathbb{R}^4$. The latter are called the 24-cell, 120-cell, and 600-cell because of their respective numbers of boundary cells. The 120-cell and 600-cell are dual to each other.

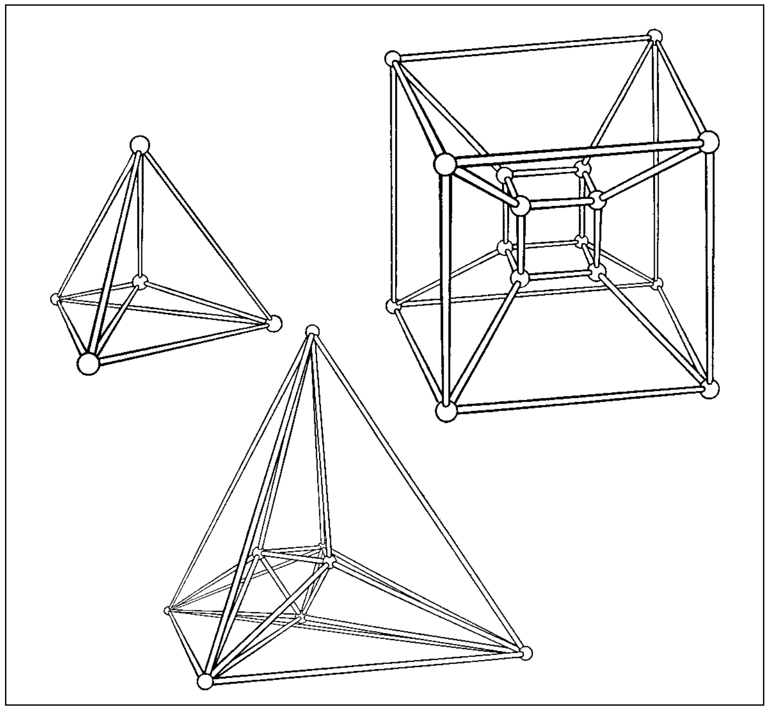

Figure 3. The three simplest regular polytopes.

- In each $\mathbb{R}^n$ there is a generalisation of the cube tiling, called the *n*-cube tiling.
  The only regular tilings other than cube tilings are two regular tilings of $\mathbb{R}^2$—the dual tilings by triangles and hexagons—and two dual tilings of $\mathbb{R}^4$, by 24-cells and 4-orthoplexes.

Schläfli's monograph contains no pictures, which may be one reason for its poor reception. The first rediscovery of the regular polytopes, by Stringham [15] in 1880, is accompanied by several illustrations, including line drawings of the 4-simplex, 4-cube, and 4-orthoplex. Today computer-generated pictures of the regular polytopes may be seen at many Web sites, often animated and/or in stereoscopic pairs. One good starting point is `http://www.ics.uci.edu/~eppstein/junkyard/polytope.html`.

However, the pictures of the four simplest regular polytopes in Hilbert and Cohn-Vossen's 1932 *Anschauliche Geometrie* [9] remain hard to beat, in my opinion. Figure 3 shows their beautifully drawn pictures of the 4-simplex, 4-cube, and 4-orthoplex, which are almost palpably 3-dimensional. And Figure 4 is their picture of the 24-cell.

Strictly speaking, of course, these pictures are projections of the vertices and edges of the polytopes onto $\mathbb{R}^2$, but we read them as frameworks in $\mathbb{R}^3$. These frameworks outline *cells* that are projectively distorted images of boundary cells of the polytopes in $\mathbb{R}^4$. The $\mathbb{R}^3$ image of the 4-simplex, for example, is a big tetrahedron with four smaller tetrahedra inside it. These five tetrahedra are the images of the five boundary tetrahedra of the 4-simplex (analogous to the common projection of a tetrahedron onto $\mathbb{R}^2$, which maps the

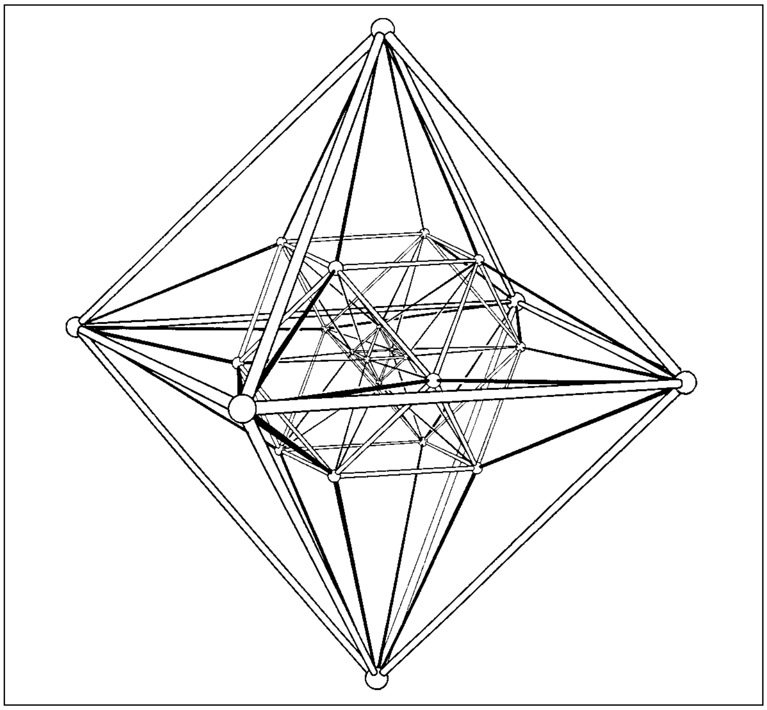

Figure 4. The 24-cell.

four boundary triangles onto one big triangle with three small ones inside it).

The other pictures similarly show:

- The 4-cube is bounded by 8 ordinary cubes, and thus it may also be called the 8-cell.
- The 4-orthoplex is bounded by 16 tetrahedra, and thus it may also be called the 16-cell.
- The 24-cell is bounded by 24 octahedra and has 24 vertices. Its *dual* polytope, which has a vertex at the centre of each boundary cell of the 24-cell, is therefore another 24-cell.

It is quite easy to find coordinates for the vertices of these polytopes. The simplest is the 4-orthoplex, whose vertices can be chosen to lie at the intersections of the unit 3-sphere in $\mathbb{R}^4$ with the coordinate axes. The 24-cell can be constructed by truncating a 4-orthoplex by hyperplanes through the midpoints of its edges and orthogonal to the coordinate axes. By taking the dual and scaling suitably, we get a 24-cell whose vertices are the 16 points

$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$$

and the 8 points

$$(\pm 1, 0, 0, 0),\ (0, \pm 1, 0, 0),\ (0, 0, \pm 1, 0),\ (0, 0, 0, \pm 1).$$

These are the 24 unit quaternions

$$\pm\frac{1}{2} \pm \frac{i}{2} \pm \frac{j}{2} \pm \frac{k}{2}, \quad \pm 1, \quad \pm i, \quad \pm j, \quad \pm k,$$

which represent (in antipodal pairs) none other than the 12 rotations of a regular tetrahedron! This is easy and fun to check, using the Gauss-Rodrigues-Cayley parameters for rotations given in the preceding section.

It seems that Steinitz [14, p. 125] was the first to notice that the vertex set of the 24-cell is the binary tetrahedral group **T**. He also recognised the vertex set of the 600-cell as the binary icosahedral group **I**. But strangely, his remarks went unnoticed until the group **I** had been through the mill of topology and combinatorial group theory. This unexpected turn in the story began in 1904 and will be recounted in the next section.

The Poincaré Homology Sphere

In 1895 Poincaré took a new approach to geometry in higher dimensions by focussing on *topology*. In his paper "Analysis situs" and five "Compléments" he introduced homology and homotopy as tools for topological classification, and he posed

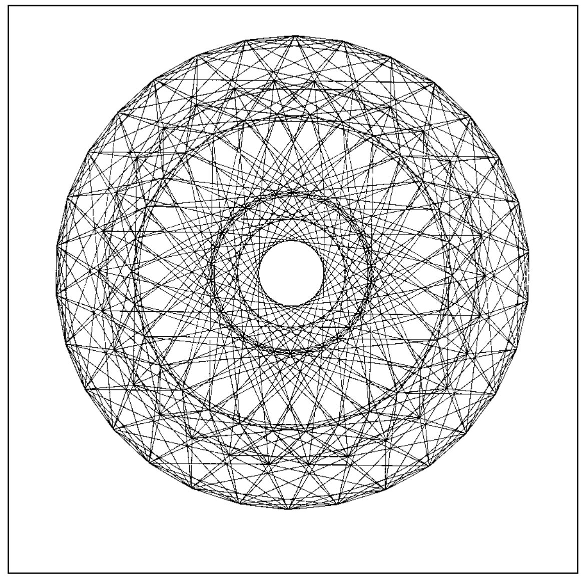

Figure 5. This van Oss projection of the 600-cell, which dates from 1901, received wide circulation as the frontispiece of Coxeter's 1948 classic *Regular Polytopes* and (redrawn) in his *Regular Complex Polytopes*.

a famous question that remains open to this day: Is every compact 3-manifold with trivial fundamental group homeomorphic to $\mathbb{S}^3$? At first he thought that any compact 3-manifold with trivial *homology* might be homeomorphic to $\mathbb{S}^3$, but this was ruled out by his discovery of the *Poincaré homology sphere* in the fifth "Complément" (1904).

He made this counterexample by gluing together two solids bounded by surfaces of genus 2, with a certain matching of canonical curves. From his construction he computed generators and relations for the fundamental group π_1, and he showed that the homology group H_1 is trivial by abelianising π_1 (though the abelianised group seems to collapse by accident). Finally, he proved that the constructed π_1 is *not* trivial by showing that it has $\mathcal{I}$ as a homomorphic image. The appearance of the icosahedral group $\mathcal{I}$ at this point is a complete surprise, because Poincaré's construction does not have any obvious symmetry.

In 1910 Dehn gave a new homology sphere construction, which introduced the *surgery* technique (see [6, p. 116]). By cutting a trefoil-knotted solid torus out of $\mathbb{S}^3$ and "sewing it back differently", he was able to improve on Poincaré's construction in two respects. The group H_1 was obviously trivial by properties of the knot, and π_1 could be explicitly identified as $\mathbf{I}$, the *binary* icosahedral group. Since $\mathcal{I}$ is a homomorphic image of $\mathbf{I}$, it was conceivable that Dehn's homology sphere was the same as Poincaré's, but this fact was established by Weber and Seifert [20] only in 1933.

In 1929 Hellmuth Kneser [11] revived interest in the Poincaré homology sphere by showing that Dehn's version of it could be constructed by identifying opposite faces of a dodecahedron. Threlfall and Seifert [19] revamped this idea by working in the universal cover $\mathbb{S}^3$ of the homology sphere. There one sees a tiling by 120 congruent dodecahedral cells, each being a fundamental region for the action of $\mathbf{I}$. This sounds suspiciously like the tiling projected onto $\mathbb{S}^3$ from the centre of the 120-cell in $\mathbb{R}^4$! Indeed it is, so Threlfall and Seifert were giving a geometric construction of the Poincaré homology sphere—as the quotient of $\mathbb{S}^3$ by $\mathbf{I}$—and a new interpretation of the 120-cell as its universal cover.

However, they did not at first know whether their subdivision of $\mathbb{S}^3$ into 120 cells was the projection of *the* 120-cell. The missing piece of the puzzle was Steinitz's 1916 remark that $\mathbf{I}$ is the vertex set of the 600-cell. Threlfall evidently saw this in 1932 and put two and two together. In a short paper [18] he improved on Steinitz by showing that the vertices and edges of the 600-cell form a *group diagram* of $\mathbf{I}$. The diagram has a vertex for each element $g \in \mathbf{I}$ and, for each g_i in a certain set of six generators, an edge from vertex g to vertex gg_i. In the dual 120-cell the twelve neighbours of the dodecahedral cell g represent the twelve neighbouring elements $gg_i^{\pm 1}$ of $g \in \mathbf{I}$—exactly as in the universal cover of the homology sphere.

The 120-cell can thus be regarded as the picture of $\mathbf{I}$ for Threlfall's generators $g_1, g_2, \ldots, g_6$, just as Figure 2 is the picture of $\mathcal{I}$ for Hamilton's generators ι and κ.

Threlfall's result brought the idea of $\mathbf{I}$ as the vertex set of the 600-cell back into mathematical consciousness, and it was developed further by Coxeter [5]. The quaternion generators of $\mathbf{I}$ give nice symmetrical coordinates for the vertices of the 600-cell. Like the vertices of the icosahedron whose symmetries they represent, they can be described quite simply and symmetrically in terms of the golden ratio $\tau = (1+\sqrt{5})/2$. Thus we not only have a construction of the 600-cell and the 120-cell, but one that Euclid would have endorsed, involving only constructible numbers.

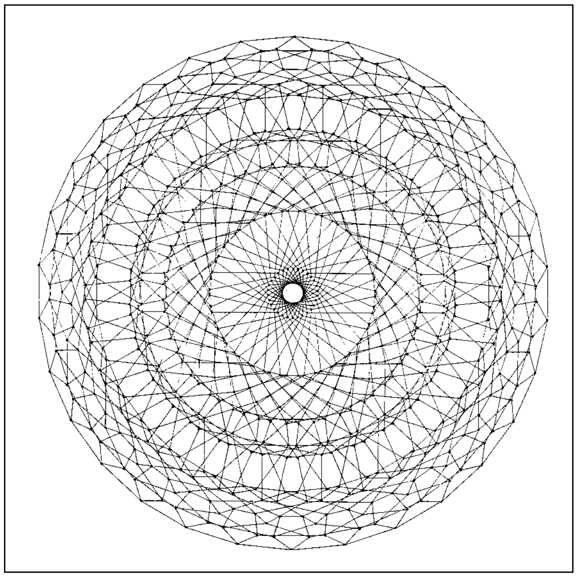

Figure 6. Chilton's drawing of the 120-cell. H. S. M. Coxeter wrote me about this projection: "When I visited van Oss around 1930, he showed me a faint pencil drawing of the 120-cell by Wythoff, made long ago, but it was too faint for reproduction. Of course it was the same as Chilton's."

Before leaving the subject of topology, we should mention that quaternion multiplication makes the sphere $\mathbb{S}^3$ of unit quaternions into a continuous group, since the product and inverse operations are obviously continuous functions of position in $\mathbb{S}^3$. Similarly, the sphere $\mathbb{S}^1$ of unit complex numbers is a continuous group under multiplication of complex numbers. In 1933 Élie Cartan showed that no other sphere has a continuous group structure, so $\mathbb{S}^3$ and $\mathbb{S}^1$ are exceptional in this respect. One wonders whether they owe their exceptional position to the exceptional algebras $\mathbb{C}$ and $\mathbb{H}$…or is it the other way around?

The 120-Cell

Despite the dual relationship between the 600-cell and the 120-cell, the latter seems more difficult to draw. The pictures of it in Coxeter's 1948 book are photographs of wire models made by Paul Donchian, and the first published drawing that I am aware of is Figure 6 by B. L. Chilton, which appeared in Coxeter's 1961 book [3].

This picture is an ordinary projection from $\mathbb{R}^4$ to $\mathbb{R}^2$, as may be guessed from its straight edges. It is a very beautiful graph, but also very flat: one can see pentagons, but it is hard to see dodecahedra, let alone combine them into a mental image of a projection of the 120-cell into $\mathbb{R}^3$.

A more convincingly "3-dimensional" image of the 120-cell is obtainable by projection onto $\mathbb{S}^3$ and then stereographic projection onto $\mathbb{R}^3$. Projecting the 120-cell from its centre onto the $\mathbb{S}^3$

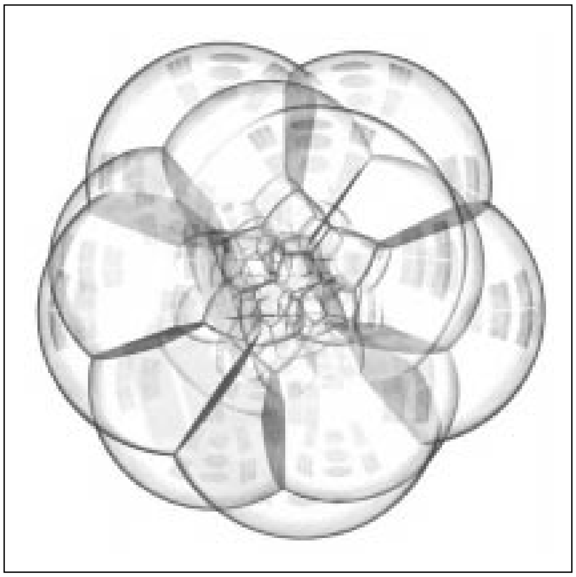

Figure 7. Soap bubble 120-cell.

containing its vertices gives a tiling of $\mathbb{S}^3$ by congruent regular spherical dodecahedra.

This tiling has the same combinatorial structure as the 120-cell, but its dodecahedral cells are slightly "inflated". Their faces are portions of great spheres in $\mathbb{S}^3$ (the analogues of great circles in $\mathbb{S}^2$), and since three cells meet along each edge, their dihedral angles are 120°. (The dihedral angles of a Euclidean dodecahedron are about $116^\circ 34'$, so the "inflation" is very slight.) Also, four cells meet at each vertex, corresponding to the four faces of the dual tetrahedron in the 600-cell.

When this 120-cell tiling of $\mathbb{S}^3$ is stereographically projected onto $\mathbb{R}^3$, spheres go to spheres and angles are preserved by the basic properties of stereographic projection. We therefore get a partition of $\mathbb{R}^3$ into 120 regions bounded by portions of spheres. Moreover, the spherical surfaces meet at 120° along each edge, and four of them meet at each vertex.

This is exactly what soap bubbles do! At least, all observed clusters of soap bubbles behave this way, and their mathematical models do also, by a deep theorem of Jean Taylor [17], proved only in 1976. Thus in principle it is possible to model the 120-cell by a soap bubble cluster! And thanks to computer graphics, it is not necessary to work with actual soap. John M. Sullivan has created beautiful simulations of a soap bubble 120-cell, shown in Figure 7 and on the cover.

The mathematical and programming details may be found in Sullivan's 1991 article [16]. Almost exactly one hundred years after the classic Fricke and Klein pictures, mathematicians now have the tools to produce a 3-dimensional sequel.

Acknowledgments

I thank John Sullivan of the University of Illinois for permission to reproduce his pictures of the 120-cell and H. S. M. Coxeter for historical information and permission to use pictures from his books. Finally, I thank Abe Shenitzer and the referees for many helpful suggestions.

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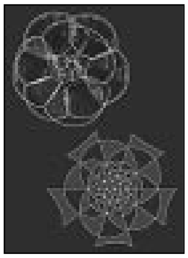
About the Cover



The cover exhibits two views, made about a century apart, of polyhedra drawn in stereographic projection. The one at the upper left was produced by John Sullivan, and is a variant of a picture he made for the cover of *The Mathematica Journal* for Winter, 1991. It shows a stereographic image in 3D of the tessellation of the 3-sphere associated to the 120-cell discussed in Stillwell's article. We know a great deal about how it was produced because Sullivan's article in *Mathematica Journal* explained the process. The projection has the geometry of a bubble cluster, and he wrote his own special soap-film shader for use with the software tool Renderman, from Pixar Animation Studios.

The other image first appeared in the book *Vorlesungen über die Theorie der Elliptischen Modulfunktionen* by Felix Klein and Robert Fricke (1890). It is a stereographic view of the tessellation of the 2-sphere associated to the dodecahedron, which is the 3D analogue of the 120-cell. The immediate source of the picture is a scan made by Mark Goresky from a copy held by the Princeton University Library. As to how the figures in the book were originally produced, we know very little. Asked about this, John Stillwell replied, "I don't know much for certain about those pictures, and I'm afraid I don't know anyone who does. The legend is that Klein arranged for engineering students to make the pictures. It is possible to construct the necessary circles in the planar pictures by ruler and compass construction. An article by Chaim Goodman-Strauss in the January 2001 issue of the *American Mathematical Monthly* reconstructs the method. The shading by parallel lines (not quite visible on the cover) was perhaps done by some engraving tool, and perhaps the shading on the picture of the spherical tiling was also. The spherical tiling impresses me the most actually, because the ellipses and shading seem to be very accurate. If these pictures really were done by engineering students, I marvel at the training they must have received!"

—*Bill Casselman* (covers@ams.org)